CMST 17(1) 87-92 (2011) DOI:10.12921/cmst.2011.17.01.87-92

COMPUTATIONAL METHODS IN SCIENCE AND TECHNOLOGY **17(1-2)**, 87-92 (2011)

The Skewes Number for Twin Primes: Counting Sign Changes of $\pi_2(x) - C_2\mathrm{Li}_2(x)$

Marek Wolf

Group of Mathematical Methods in Physics
University of Wrocław
Pl. Maxa Borna 9, PL-50-204 Wrocław, Poland
e-mail: mwolf@ift.uni.wroc.pl

(Received: 05 April 2011; revised: 12 June 2011; accepted: 15 August 2011; published online: 5 October 2011)

Abstract: The results of computer investigation of the sign changes of the difference between the number of twin primes $\pi_2(x)$ and the Hardy-Littlewood conjecture $C_2\mathrm{Li}_2(x)$ are reported. It turns out that $d_2(x) = \pi_2(x) - C_2\mathrm{Li}_2(x)$ changes the sign at unexpectedly low values of x and for $x < 2^{48} = 2.81... \times 10^{14}$ there are 477118 sign changes of this difference. It is conjectured that the number of sign changes of $d_2(x)$ for $x \in (1, T)$ is given by $\sqrt{T}/\log(T)$. The running logarithmic densities of the sets for which $d_2(x) > 0$ and $d_2(x) < 0$ are plotted for x up to 2^{48}.

Key words: primes, twins, Skewes number

I. INTRODUCTION

Let $\pi(x)$ be the number of primes smaller than x and let Li(x) denote the logarithmic integral:

$$\mathrm{Li}(x) = \int_2^x \frac{du}{\log(u)}. \tag{1}$$

The Prime Number Theorem tells us that $\mathrm{Li}(x)/\pi(x)$ tends to 1 for $x \to \infty$ and the available data (see [24, Table 14, p. 175] or [8, Table 5 and 6]) show that always $\mathrm{Li}(x) > \pi(x)$. This last experimental observation was the reason for the common belief in the past that the inequality $\mathrm{Li}(x) > \pi(x)$ is generally valid. However, in 1914 J.E. Littlewood showed [20] (see also [7]) that the difference between the number of primes smaller than x and the logarithmic integral up to x changes the sign infinitely many times. The smallest value x_S that for the first time $\pi(x_s) \geq \mathrm{Li}(x_s)$ holds is called Skewes number. We have used "$\geq$" to avoid the case of integer value of $\mathrm{Li}(x_S)$, although we believe that for $n \in \mathbb{N}$ there will be $\mathrm{Li}(n) \notin \mathbb{N}$, like we know log($n$) is for $\forall n$ irrational. In 1933, assuming the truth of the Riemann hypothesis, S. Skewes [29] argued that it was certain that $d(x) := \pi(x) - \mathrm{Li}(x)$ changes sign for some $x_S = 10^{10^{10^{34}}}$. In 1955, Skewes [30] found, without assuming the Riemann hypotheses, that $d(x)$ changes sign at some

$$x_S < \exp\exp\exp\exp(7.705) < 10^{10^{10^{10^3}}}.$$

This enormous bound for x_S was reduced by Cohen and Mayhew [5] to $x_S < 10^{10^{529.7}}$ without using the Riemann hypothesis. In 1966, Lehman [19] showed that between 1.53×10^{1165} and 1.65×10^{1165} there are more than 10^{500} successive integers x for which $\pi(x) > \mathrm{Li}(x)$. Following the method of Lehman in 1987 H.J.J. te Riele [31] showed that between 6.62×10^{370} and 6.69×10^{370} there are more than 10^{180} successive integers x for which $d(x) > 0$. The lowest present day known estimation of the Skewes number is around 10^{316}, see [2] and [27].

The number of sign changes of the difference $d(x)$ for x in a given interval (1, T), which is commonly denoted by $\nu(T)$, see [7], was discussed for the first time by A.E. Ingham in 1935 [12] chapter V, [11] and next by S. Knapowski [16]. Regarding the number of sign changes of $d(x)$ in the interval (1, T), Knapowski [16] proved that

$$\nu(T) \geq e^{-35} \log\log\log\log T \tag{2}$$

provided $T \geq \exp\exp\exp\exp(35)$. Further results about $\nu(T)$ were obtained by J. Pintz [21, 22] and J. Kaczorowski [13, 14]. In particular, in [14] Kaczorowski proved that there exists such a positive constant c_3 that for sufficiently large T the inequality

$$\nu(T) \geq c_3 \log(T) \tag{3}$$

holds. In [28] J.-C. Schlage-Puchta proved, assuming the Riemann Hypothesis, that

$$\nu(T) > \frac{\log(T)}{e^{e^{16.7}}} - 1. \tag{4}$$

More general results on the sign changes can be found in the recent paper [15].

In this paper we will look for the analog of the Skewes number for twin primes, i.e. pairs of primes separated by 2: {(3, 5), (5, 7), (11, 13), ..., (59, 61), ... }.

Let us denote the number of twin prime pairs $(p, p + 2)$ with $p + 2 < x$ by $\pi_2(x)$. Then the unproved (see [25]) conjecture B of Hardy and Littlewood [9] on the number of prime pairs $p, p + d$ applied to the case $d = 2$ gives, that

$$\pi_2(x) \sim C_2 \mathrm{Li}_2(x) \equiv C_2 \int_2^x \frac{u}{\log^2(u)} du, \tag{5}$$

where C_2 is called “twin constant” and is defined by the following infinite product:

$$C_2 \equiv 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) = 1.3203236316937... \tag{6}$$

For the first time the conjecture (5) was checked computationally up to 8×10^{10} by R.P. Brent [4] who noticed the sign changes of the difference $\pi_2(x) - C_2\mathrm{Li}_2(x)$, but he neither mentioned the analogy with Skewes number nor counted these sign changes. We analyzed the difference $d_2(x) := \pi_2(x) - C_2\mathrm{Li}_2(x)$ using the computer for x up to $T = 2^{48} \approx 2.814 \times 10^{14}$. It took 195 CPU days to reach $T = 2^{48}$ on the 64 bits AMD® Opteron 2700 MHz processor.

To calculate the integral $\mathrm{Li}_2(x)$ during the main run of the program till 2^{48} we used the 10-point Gauss quadrature [23]. This integral was calculated numerically in successive intervals between consecutive twins and added to the previous value. Such a method is not very time-consuming and the number of performed arithmetical operations does not depend on x. There are also power series representations of the logarithmic integral. We use the following convention for the li(x) (here *v.p.* stands for French *valeur principale*, i.e. Cauchy principal value):

$$\mathrm{li}(x) - v.p.\int_0^x \frac{du}{\log(u)} = \lim_{\varepsilon\to 0}\left(\int_0^{1-\varepsilon} \frac{du}{\log(u)} + \int_{1+\varepsilon}^x \frac{du}{\log(u)}\right), \tag{7}$$

hence we have Li(x) = li(x) – li(2). Integration by parts gives the asymptotic expansion:

$$\mathrm{li}(x) \sim \frac{x}{\log(x)} + \frac{x}{\log^2(x)} + \frac{2x}{\log^3(x)} + \frac{6x}{\log^4(x)} + ... + \frac{n!x}{\log^{n+1}(x)+...}. \tag{8}$$

which should be cut at $n_0 = \lfloor \log(x) \rfloor$ – beginning with this index the following terms are increasing. There is a series giving li(x) for all $x > 1$ and quickly convergent which has $n!$ in denominator and $\log^n(x)$ in nominator instead of the opposite order in (8) (see [3, p. 126, Entry 14])

$$\int_\mu^x \frac{du}{\log(u)} = \gamma + \log\log(x) + \sum_{n=1}^\infty \frac{\log^n(x)}{n \cdot n!} \quad \text{for} \quad x > 1, \tag{9}$$

where γ = 0.5772156649... is the Euler-Mascheroni constant and μ = 1.451369234883381... is the Soldner constant defined by (see [3, p. 123, Eq. (11.3)])

$$\mathrm{li}(\mu) = v.p.\int_0^\mu \frac{du}{\log(u)} = 0.$$

An even faster converging series was discovered by Ramanujan [3, p. 130, Entry 16]:

$$\int_u^x \frac{du}{\log(u)} = \gamma + \log(\log(x)) + \sqrt{x}\sum_{n=1}^\infty \frac{(-1)^{n-1}(\log(x))^n}{n!2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}$$

$$\text{for } x > 1. \tag{10}$$

Because we have

$$\mathrm{Li}_2(x) = \mathrm{Li}(x) - \frac{x}{\log(x)}$$

it is possible to calculate values of $\mathrm{Li}_2(x)$ using the above series. A disadvantage of these series is that the number of operations (including the time consuming calculation of log(x)) increases with x and is larger than the number of operations needed in the numerical integration.

As for the set of all primes initially the inequality $C_2\mathrm{Li}_2(x) > \pi_2(x)$ holds, but it turns out that there are surprisingly many sign changes of $d_2(x) = \pi_2(x) - C_2\mathrm{Li}_2(x)$ for x in the interval $(1, 2^{48})$. The first sign change of $d_2(x)$ appears at the twin pair (1369391, 1369393) and up to $T = 2^{48}$ there are 477118 sign changes of $d_2(x)$. We have collected positions of all these sign changes in one file which is available for downloading from http://www.ift.uni.wroc.pl/~mwolf/Skewesy_twins.zip. Let $\nu_2(T)$ denote, by analogy with usual primes, the number of sign changes of $d_2(x)$ in the interval $(1, T)$. Table 1 contains the recorded number of sign changes of $\pi_2(x) - C_2 \mathrm{Li}_2(x)$ up to $T = 2^{21}$, 2^{22}, ... , 2^{48}. We have checked the numbers $v_2(T)$ up to $T = 2^{34} = 1.718 \times 10^{10}$ independently calculating the integral $\mathrm{Li}_2(x)$ from the series (10) and these results are presented in Table 1 in the third column and are marked with an asterisk. The first 1274 positions of sign changes of $d_2(x)$ obtained by these two methods of calculating the integral $\mathrm{Li}_2(x)$ were the same. The first difference between both methods appears at twin pairs (3067608611, 3067608613) and (3067609091, 3067609093), which were not detected using the more accurate formula (10). Next twin primes detected by the two methods are the same until the twin pairs (7809444029, 7809444031). In general, among over 9100 sign changes up to 2^{34} there were 17 differences in the positions of sign changes of $d_2(x)$ obtained by two methods of calculating the integral $\mathrm{Li}_2(x)$.

The values of T searched by the direct checking are of small magnitude from the point of view of mathematics, but large for modern computers.

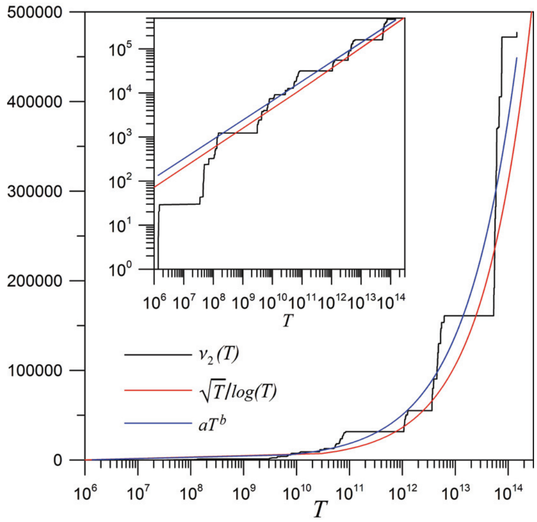

Fig. 1. The plot showing the comparison of the actual values of $v_2(T)$ found by a computer search with the conjecture (11). There are 10 crossing of the function $v_2(T)$ and $\sqrt{T}/\log(T)$ in this plot up to 2^{48}. All 477118 sign changes of $d_2(x)$ are plotted. In the inset plot on the double logarithmic scale is presented

The observed numbers $v_2(T)$ behave somewhat erratically (see Fig. 1), in particular there are large gaps without any change of sign of the $d_2(x)$. If one assumes the power-like dependence of $v_2(T)$ then the fit by the least square method gives the function aT^b, where $a = 0.2723$ ... and $b = 0.4389$ ... . Instead of such accidentally looking parameters of the pure power-like dependence we suggest the function $\sqrt{T}/\log(T)$ as an approximation to $v_2(T)$ – it is a more natural function, without any free parameters and taking values very close to the least square fit aT^b, see Fig. 1. Thus we state the following conjecture:

$$v_2(T) \sim \sqrt{T}/\log(T). \tag{11}$$

We have picked out function $\sqrt{T}/\log(T)$ after several trials and we are not able to give even heuristic arguments in favour of it. The conjecture (11) is supported by the fact that there are 10 crossings of the curve $\sqrt{T}/\log(T)$ with the staircase-like plot of $v_2(T)$ obtained directly from the computer data. The last column in Table 1 contains the values of the function $\sqrt{T}/\log(T)$. If the conjecture (11) is true, then there is infinity of twins.

Table 1. The number of sign changes of $d_2(x)$

T	$v_2(T)$	$v_2(T)$ (*)	$\sqrt{T}/\log(T)$	T	$v_2(T)$	$\sqrt{T}/\log(T)$
2^{21}	29	29	99	2^{35}	12682	7641
2^{22}	29	29	134	2^{36}	23634	10505
2^{23}	29	29	182	2^{37}	31641	14455
2^{24}	29	29	246	2^{38}	31641	19905
2^{25}	29	29	334	2^{39}	31641	27428
2^{26}	238	238	455	2^{40}	38899	37819
2^{27}	854	854	619	2^{41}	55106	52180
2^{28}	1226	1226	844	2^{42}	90355	72037
2^{29}	1226	1226	1153	2^{43}	161031	99506
2^{30}	1226	1226	1576	2^{44}	161031	137525
2^{31}	1226	1226	2157	2^{45}	161031	190168
2^{32}	2854	2852	2955	2^{46}	405289	263091
2^{33}	7383	7381	4052	2^{47}	472000	364151
2^{34}	9115	9113	5562	2^{48}	477118	504258

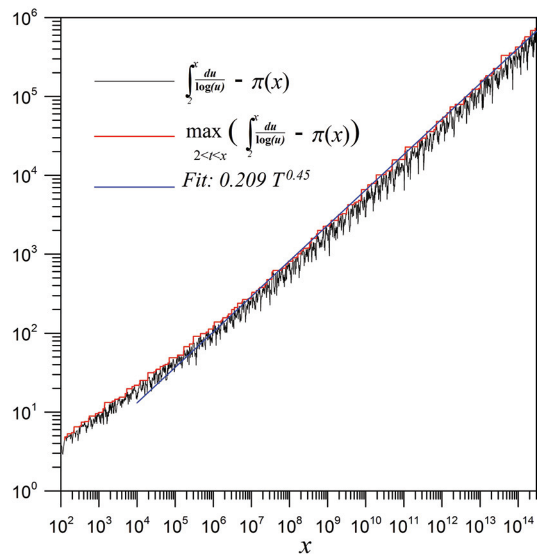

Fig. 2. The plot of $d(x)$ and error term $\Delta(x)$. The power fit was made for $10^6 < x < 2^{48}$. The first crossing of the axis x will appear around 10^{316}

It seems to be very difficult to gain some analytical insight to why there are so many sign changes of $\pi_2(x)-C_2\mathrm{Li}_2(x)$. As (5) is not proved, hence error term for it is also not known (for heuristic approximate formula for averages of the remainders in the Hardy-Littlewood conjecture B see [18]). The best error term for the Prime Number Theorem under the Riemann Hypothesis is $|\pi(x)-\mathrm{Li}(x)|=\mathcal{O}\left(\sqrt{x}\log(x)\right)$. In Fig. 2 we present the computer data for two functions: the running difference $d(x)=\mathrm{Li}(x)-\pi(x)$ and the error term:

$$\Delta(x)=\max_{2<t<x}|\pi(t)-\mathrm{Li}(t)|. \qquad (12)$$

Characteristic oscillations of $d(x)$ are fully described by the explicit formula for $\pi(x)$, see e.g. [8, formula (3) and Fig. 4]. In Fig. 3 $|d_2(x)|$ and the error term

$$\Delta_2(x)=\max_{2<t<x}|\pi_2(t)-C_2\mathrm{Li}_2(t)| \qquad (13)$$

is plotted for $x<2^{48}$. As it is seen from these figures, the behavior of $d(x)$ and $d_2(x)$ is completely different with rapid oscillations of $d_2(x)$ of many orders. However, the functions $\Delta(x)$ and $\Delta_2(x)$ are quite similar: the error term for twins $\Delta_2(x)$ is smaller than $\Delta(x)$ but the difference is not significant: the power-like fits to $\Delta(x)$ and $\Delta_2(x)$ give:

$$\alpha x^{\beta}, \quad \alpha=0.209\ldots, \quad \beta=0.45\ldots \quad \text{for} \quad \Delta(x) \qquad (14)$$

$$\alpha_2 x^{\beta_2}, \quad \alpha_2=0.337\ldots, \quad \beta_2=0.418\ldots \quad \text{for} \quad \Delta_2(x). \qquad (15)$$

Here the slopes $\beta\approx\beta_2$ and prefactors α and α_2 are very close. Thus it seems that the sizes of the error terms do not account for an enormous difference in the value of the Skewes number. In fact all considerations of Skewes, Kaczorowski and others were based on the existence of explicit formulas and there are no analogs of explicit formulas for twins. However, Turan [33] introduced the following Dirichlet series with the aim to study twins:

$$T(s):=\sum_{n>3}\frac{\Lambda(n-1)\Lambda(n+1)}{n^s} \quad (\Re\mathrm{e}\ s>1), \qquad (16)$$

where $\Lambda(n)$ is the von Mangoldt function:

$$\Lambda(n)=\begin{cases} 0 & \text{if } n=1 \\ \log p & \text{if } n=p^k \text{ for some prime } p \text{ and integer } k\geq 1, \\ 0 & \text{if } n \text{ has at least two different prime factors.}\end{cases} \qquad (17)$$

In 2004, in a preprint publication [1] Arenstorf attempted to prove that there are infinitely many twins. Arenstorf tried to continue analytically $T(s)-C_2/(s-1)$ to $\Re\mathrm{e}\, s=1$, but shortly after an error in the proof was pointed out by Tenenbaum [32]. For recent progress in the direction of the proof of the infinite number of twins see [17].

The comparison of Figures 2 and 3 shows that $\pi_2(x)\sim C_2\mathrm{Li}_2(x)$ is better than $\pi(x)\sim\mathrm{Li}(x)$ in the sense that there are almost half a million points where $d_2(x)$ is zero in the Fig. 3 while in Fig. 2 there are no crossings of x axis at all. This observation can be quantifying with the notion of the logarithmic density. In [26] it was proposed to use the logarithmic density to measure the different biases in the distribution of prime numbers. In particular, for the case of the sign changes of $d(x)$ it was shown that the logarithmic density of the set $\{x:\mathrm{Li}(x)<\pi(x)\}$ defined by

$$\delta_{\{x:\mathrm{Li}(x)<\pi(x)\}}=\lim_{x\to\infty}\frac{1}{\log(x)}\sum_{\substack{2\leq n<x\\ \mathrm{Li}(n)<\pi(n)}}\frac{1}{n} \qquad (18)$$

is equal to $\delta_{\{x:\mathrm{Li}(x)<\pi(x)\}}=2.7\ldots\times 10^{-7}$. Hence the inequality $\mathrm{Li}(x)<\pi(x)$ holds on the set of (precisely defined) negligible measure. Here we will define two logarithmic densities for twin primes as follows:

$$\delta_{+}=\lim_{x\to\infty}\frac{1}{\log x}\sum_{\substack{2\leq n<x\\ d_2(n)>0}}\frac{1}{n} \qquad (19)$$

$$\delta_{-}=\lim_{x\to\infty}\frac{1}{\log x}\sum_{\substack{2\leq n<x\\ d_2(n)<0}}\frac{1}{n}. \qquad (20)$$

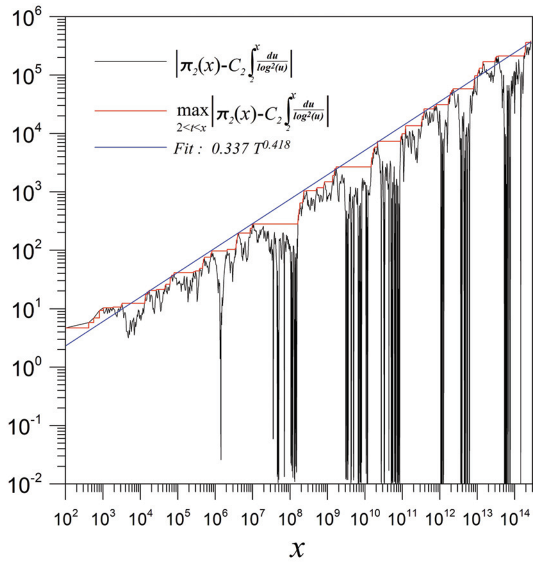

Fig. 3. The plot of $|d_2(x)|$ and error term $\Delta_2(x)$. Sign changes of the $d_2(x)$ and values smaller than 10^{-2} were artificially set to 10^{-2}. In blue the power-like fit $0.337\times x^{0.418}$ to $\Delta_2(x)$ obtained by the least-square method is plotted

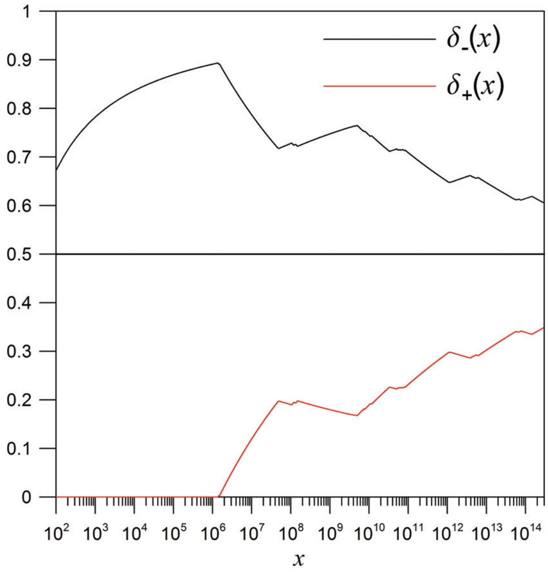

Fig. 4. The plots of the running logarithmic densities $\delta_+(x)$, $\delta_-(x)$ defined in the text. Each plot consists of 28025 points: the values of $\delta(x)$'s were recorded at the progression $x = 100 \times (1.001)^n$

We do not have at our disposal any formulas like those in [26] and we have to turn to the brute force numerical calculation of finite size approximations $\delta_+(x)$ and $\delta_-(x)$ given by expressions (19) and (20) without limit operation $\lim_{x\to\infty}$. In these computations we have used positions of all sign changes collected earlier. The resulting running logarithmic densities are plotted in Fig. 4. The sum for $\delta_-(x)$ starts from 1/5, because 5 is the end of the first twin primes pair. It is the reason why the plot of $\delta_-(x)$ in Fig. 4 starts from about 0.67. Up to $x = 2^{31}$ the data for Fig. 4 was obtained by direct summing of the harmonic sums, for $x > 2^{31} \approx 2.15 \times 10^9$ the incredible accurate approximation [6, 10, pp. 76-78]:

$$\sum_{k=n}^{m} \frac{1}{k} = \log\left(m+\frac{1}{2}\right) - \log\left(n-\frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{n^2}\right) \qquad (21)$$

was used (the implied in $\mathcal{O}$ constant is much smaller than 1). For $n \approx 10^9$ the error made by using the above formula is of the order 10^{-18}. To calculate the harmonic series up to $x = 2.8 \times 10^{14}$ directly by adding all numbers $1/n$ would take from one to a few months of CPU time, depending on the processor. The plots presented in Fig. 4 suggest the following conjecture

$$\delta_+ = \delta_- = \frac{1}{2}. \qquad (22)$$

The difference of many hundreds of orders between values of x such that $\pi(x) - \mathrm{Li}(x)$ and $\pi_2(x) - C_2\mathrm{Li}_2(x)$ changes the sign for the first time is astonishing. We can give an example from physics. Let us make the mapping: sign changes of $d(x)$ correspond to energy levels of hydrogen and sign changes of $d_2(x)$ correspond to the spectrum of helium. Then ground states of hydrogen and of helium will correspond to x_S and the first sign change of $d_2(x)$ accordingly. The experiments show that the energies of the ground states of the hydrogen and helium are –13.6 eV and –79 eV respectively and do not differ by hundreds of orders.

Acknowledgements

We would like to thank Prof. A. Jadczyk for reading the manuscript and for helpful comments. The author is also very thankful to the referees for their valuable remarks and suggestions.

References

[1] R.F. Arenstorf, There are infinitely many prime twins, 26-th May 2004. http: //arxiv.org/abs/math/0405509v1.

[2] C. Bays, R.H. Hudson, *A new bound for the smallest x with* $\pi(x) - \mathrm{Li}(x)$. Mathematics of Computation 69: 1285-1296, (2000). available from http://www.ams. org/mcom/2000-69-231/S0025-5718-99-01104-7/S0025-5718-99-01104-7.pdf.

[3] B.C. Berndt, *Ramanujan's Notebooks*, Part IV. Springer Verlag (1994).

[4] R.P. Brent, *Irregularities in the distribution of primes and twin primes.* Mathematics of Computation 29: 43-56 (1975). available from http://wwwmaths.anu.edu.au/ ~brent/pd/rpb024.pdf.

[5] A.M. Cohen, M.J.E. Mayhew, *On the difference* $\pi(x) - \mathrm{Li}(x)$. Proc. London Math. Soc. 18: 691-713 (1968).

[6] D.W. DeTemple, *A quicker convergence to Euler's constant.* The American Mathematical Monthly 100 (5): 468-470 (1993).

[7] W. Ellison, F. Ellison, *Prime Numbers*. John Wiley and Son (1985).

[8] A. Granville, G. Martin, *Prime number races.* American Mathematical Monthly 113: 1-33 (2006).

[9] G.H. Hardy, J.E. Littlewood, *Some problems of 'Partitio Numerorum' III: On the expression of a number as a sum of primes*. Acta Mathematica 44: 1-70 (1922).

[10] J. Havil, *Gamma: Exploring Euler's Constant*. Princeton University Press, Princeton, NJ (2003).

[11] A.E. Ingham, *A note on the distribution of primes*. Acta Arithmetica I: 201-211, 1936. available from http://matwbn.icm.edu.pl/ksiazki/aa/aa1/aa1116.pdf.

[12] A.E. Ingham, *The distribution of prime numbers*. unchanged reprint: Hafner Publ. Comp. (New York) (1971).

[13] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula. I.* Acta Arithetica 44: 365-377 (1984). available from http://matwbn.icm.edu.pl/ ksiazki/aa/aa44/aa4446.pdf.

[14] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula. II.* Acta Arithetica 45: 65-74 (1984). available from http://matwbn.icm.edu.pl/ ksiazki/aa/aa45/aa4517.pdf.

[15] J. Kaczorowski, K. Wiertelak, *Oscillations of a given size of some arithmetic error terms.* Trans. Amer. Math. Soc. 361: 5023-5039 (2009).
[16] S. Knapowski, *On sign changes of the difference* $\pi(x)$ $-\mathrm{Li}(x)$. Acta Arithmetica VII: 106-119 (1962).
[17] J. Koreevar, *Distributional Wiener-Ikehara theorem and twin primes.* Indag. Mathem., N.S. 16: 3749, 2005. Available from http://staff.science.uva.nl/~korevaar/DisWieIke.pdf.
[18] J. Korevaar, H. te Riele. *Average prime-pair counting formula*. Math. Comput. 79 (270): 1209-1229 (2010).
[19] R.S. Lehman, *On the difference* $\pi(x) - \mathrm{Li}(x)$. Acta Arithmetica XI: 397-410 (1966). http://matwbn.icm.edu.pl/ksiazki/aa/aa11/aa11132.pdf.
[20] J.E. Littlewood, *Sur la distribution des nombres premieres*. Comptes Rendus 158: 1869-1872 (1914).
[21] J. Pintz, *On the remainder term of the prime number formula. III*. Studia Sci. Math. Hungar. 12: 343-369 (1977).
[22] J. Pintz, On the remainder term of the prime number formula. IV. Studia Sci. Math. Hungar. 13: 29-42 (1978).
[23] H.W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, *Numerical Recipes: The Art of Scientific Computing.* Cambridge University Press, New York, NY (1986).
[24] P. Ribenboim, *The Little Book of Big Primes*. 2ed., Springer (2004).
[25] M. Rubinstein, *A simple heuristic proof of Hardy and Littlewood conjecture B*. Amer. Math. Monthly, 100: 456-460 (1993).
[26] M. Rubinstein, P. Sarnak, *Chebyshevs bias*. Experimental Mathematics 3: 173-197 (1994).
[27] Y. Saouter, P. Demichel, *A sharp region where* $\pi(x) - \mathrm{li}(x)$. *is positive*. Math. Comput. 79 (272): 2395-2405 (2010).
[28] J.C. Schlage-Puchta, *Sign changes of* $\pi(x,q,1) - \pi(x,q,a)$. Acta Mathematica Hungarica, 102: 305-320 (2004).
[29] S. Skewes, *On the difference* $\pi(x) - \mathrm{Li}(x)$. J. London Math. Soc. 8: 277-283, 1934. available from http://www.ift.uni.wroc.pl/~mwolf/Skewes1933.pdf.
[30] S. Skewes. *On the difference* $\pi(x) - \mathrm{Li}(x)$. II. Proc. London Math. Soc. 5: 48-70, 1955. available from http://www.ift.uni.wroc.pl/~mwolf/Skewes1955.pdf. 12
[31] H.J. te Riele, *On the difference* $\pi(x) - \mathrm{Li}(x)$. Mathematics of Computation 48: 323-328 (1987).
[32] G. Tenenbaum, *Re: Arenstorf's paper on the twin prime conjecture.* NM- BRTHRY@listserv.nodak.edu mailing list. 8 Jun 2004. http://listserv.nodak. edu/cgi-bin/wa.exe? A2=ind0406&L=nmbrthry&F=&S=&P=1119.
[33] P. Turan, *On the twin-prime problem II.* Acta Arithmetica XIII: 61-89, 1967. available from http://matwbn.icm.edu.pl/ksiazki/aa/aa13/aa1315.pdf. 13

Marek Wolf is a member of the Group of Mathematical Methods in Physics at the Institute of Theoretical Physics, University of Wroclaw. His M.Sc. in Physics (1978) thesis was devoted to the Nambu-Goto string, Ph.D. in Physics (1982) was about Central Charges in the Supersymmetric Quantum Field Theory. Since 1984, he has been doing computer experiments in physics and mathematics. He was habilitated in 1993.