CMST SI(2) 77-96 (2010)

DOI:10.12921/cmst.2010.SI.02.77-96

COMPUTATIONAL METHODS IN SCIENCE AND TECHNOLOGY **Special Issue 2010,** XX-XX

Propagation of Ultrashort Pulses in a Nonlinear Medium

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(Received: 29 June 2010; accepted: 13 September 2010)

Abstract: In this paper, using a general propagation equation of ultrashort pulses in an arbitrary dispersive nonlinear medium derived in [8] we study in the specific case of Kerr media. An obtained ultrashort pulse propagation equation which is called Generalized Nonlinear Schrödinger Equation usually has a very complicated form and looking for its solutions is usually a "mission impossible". Theoretical methods to solve this equation are effective only for some special cases. As an example we describe the method of a developed elliptic Jacobi function expansion. Several numerical methods of finding approximate solutions are simultaneously used. We focus mainly on the following methods: Split-Step, Runge-Kutta and Imaginary-time algorithms. Some numerical experiments are implemented for soliton propagation and interacting high order solitons. We consider also an interesting phenomenon: the collapse of solitons.

Key words: ultrashort pulses, Kerr media, Generalized Nonlinear Schrödinger Equation, solitons

I. INTRODUCTION

Creation and propagation of ultrashort laser pulses (in fs) in a medium has been intensively researched (both theoretically and experimentally) on the course of the last few years [1, 3, 17, 20]. Modern lasers can generate pulses as short as a few optical cycles, with durations on the order of 10^{-15} seconds. The short duration of these pulses allows us to look at very fast events, such as molecules vibrating, or charge transfer in biological systems. One can also manipulate the shape of the pulse and use it to control precisely the quantum phenomena, such as the formation of molecules from cold atoms (noncrystalline structure), or the initiation of a quantum phase transition in a solid. The ultrashort pulse could be used as a photonic reagent in different chemical reactions. Short pulse with a large energy focused by lens gives us a very high peak intensity which leads to several potential applications as in creation of unusual states of matter (plasmas) by reaching very high temperatures, or using it as an energy source for X-ray lasers.

During the propagation of ultrashort pulses in the medium, several new effects have been observed in the comparison with the propagation process of short pulses (in ps), namely the effects of dispersion and nonlinear effects of higher orders. Under the influence of these effects, we have complicated changes both in amplitude and spectrum of the pulse. It splits into constituents and its spectrum also evolves into several bands which are known as optical shock and self-frequency shift phenomena [1, 3, 11, 16]. These effects should be studied in detail for future concrete applications of ultrashort pulses, especially in the domain of optical soliton communication.

We have recently developed a powerful method of deriving a general equation for short-duration intense pulses [8, 14, 15]. This method is based on a consistent and mathematically rigorous expansion of the nonlinear wave equation, which treats the nonlinear processes involved in the problem as perturbations. Using this method for the Kerr medium in the consideration of the delayed nonlinear response of the medium, induced by the stimulated Raman scattering and the characteristic features of both the spectrum and the intensity of the pulse, in Sec. II we will obtain an approximate equation in the most condensed form, which describes the propagation of the ultrashort pulses, called the generalized nonlinear Schrödinger equation (GNLS). In the general case it is very difficult to find analytic solutions for this equation. A review of analytic methods is given in [5]. We consider a normalized form of this equation and demonstrate its general features.

We will analyze in detail the influence of the third-order dispersion (TOD), the self-steepening and the self-shift frequency for the ultrashort pulses in some special cases. When the higher-order terms are included, the pulse propagation equation becomes very complicated [9]. Under some conditions its solutions in the form of dark and bright solitons are obtained [12]. We will use the developed Jacobi expansion formalism introduced there in finding analytic solutions for the case when the fourth-order dispersion (FOD) is also included. But generally we should use different numerical methods to solve it. In Sec. III we present three efficient methods, the Split-Step Fourier, the fourth order Runge-Kutta and the imaginary-time methods. We investigate a very interesting phenomenon in Sec. IV, namely the collapse of solitons. Sec. V contains conclusions.

II. PROPAGATION EQUATION FOR ULTRASHORT PULSES

II.1. General pulse propagation equation in the nonlinear dispersion medium

The Maxwell equations can be used to obtain the following nonlinear wave equation for the electric field [1, 2, 4, 15]:

$$\nabla^2\vec{E}(\vec{r},t)-\nabla\left(\nabla\vec{E}(\vec{r},t)\right)-\frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}= \\ =\mu_0\frac{\partial^2\vec{P}_l(\vec{r},t)}{\partial t^2}+\mu_0\frac{\partial^2\vec{P}_{nl}(\vec{r},t)}{\partial t^2}, \tag{1}$$

where $\vec{P}_l(\vec{r},t)$ and $\vec{P}_{nl}(\vec{r},t)$ are the linear and nonlinear polarization, respectively.

The electric field $\vec{E}$ is treated as a superposition of monochromatic constituents with different frequencies and wavevectors centered at their central values ω_0 and $\vec{k}_0$. We limit ourself only to considering the propagation of the electric field in an arbitrary direction, say Oz (usually chosen as the direction of $\vec{k}_0$), so we can write

$$\vec{E}(r,t)=\vec{x}E(z,t)= \\ =\frac{1}{2}\vec{x}\left[A(z,t)e^{-i\omega_0 t+ik_0 z}+c.c\right], \tag{2}$$

where $\vec{x}$ is the unit vector of the x axis perpendicular to the propagation direction, $A(z,t)$ is the complex envelope function, $c.c$ denotes the complex conjugate of the first term.

For the homogeneous isotropic medium the linear polarization vector of the medium is expressed as follows

$$\vec{P}_l(\vec{r},t)\equiv\vec{P}_l(\vec{z},t)=\vec{x}P_l(z,t)= \\ =\vec{x}\varepsilon_0\int_{-\infty}^{+\infty}\chi^{(1)}(t-t')E(z,t')dt'=\vec{x}\varepsilon_0\tilde{\chi}^{(1)}*E, \tag{3}$$

where * denotes the convolution product which displays the causality: the response of the medium in time t is caused by the action of the electric field in all previous times t'. The quantity $\chi^{(1)}$ is the susceptibility of the medium. It is a scalar.

The nonlinear polarization vector is generally expressed as follows

$$\vec{P}_{nl}(\vec{r},t)=\varepsilon_0\Bigg[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\chi^{(2)}(t-t_1,t-t_2):\vec{E}(\vec{r},t_1)\vec{E}(\vec{r},t_2)dt_1dt_2+ \\ +\iiint_{-\infty}^{+\infty}\chi^{(3)}(t-t_1,t-t_2,t-t_3)\vdots\vec{E}(\vec{r},t_1)\vec{E}(\vec{r},t_2)\vec{E}(\vec{r},t_3)dt_1dt_2dt_3+\ldots\Bigg] \tag{4}$$

where $\chi^n(t-t_1,t-t_2,\ldots,t-t_n)$ is the n-order nonlinear susceptibility. For the homogeneous isotropic medium, because of the spatial inversion symmetry the elements of the even-order nonlinear susceptibility $\chi^{2k}(t-t_1,\ldots,t-t_{2k})$ disappear [1, 2, 4]. In the expression (5) we have only the nonlinear polarizations of odd orders. We consider in detail only the third-order nonlinear susceptibility (the Kerr medium). Then the tensor $\chi^{(3)}$ has 3^4 = 81 elements (as a matrix with 3 lines and 27 columns), but only 21 of its elements are different from zero and three are independent [1]. We have therefore

$$\vec{P}_{nl}(r,t)\equiv\vec{P}_{nl}(z,t)=\vec{x}P_{nl}(z,t)=\vec{x}.\varepsilon_0\,\times \\ \times\iiint_{-\infty}^{+\infty}\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3)E(z,t_1)E(z,t_2)E(z,t_3)dt_1dt_2dt_3. \tag{5}$$

In the hierarchy of the magnitudes, the nonlinear polarization is much smaller than the electric field and the linear polarization

$$\left|\vec{P}_{nl}(z,t)\right|\ll\left|\vec{P}_l(z,t)\right|,\quad\left|\vec{P}_{nl}(z,t)\right|\ll\varepsilon_0\left|\vec{E}(z,t)\right|,$$

so it can be considered as a perturbation and we have the approximate formula [15]: $\nabla.\vec{E}(z,t)\approx 0$. Substituting these results into (1) we obtain the following scalar wave equation

$$\Delta E(z,t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E(z,t)+\tilde{\chi}^{(1)}*E\right)= \\ =\frac{1}{\varepsilon_0c^2}\frac{\partial^2P_{nl}(z,t)}{\partial t^2}. \tag{6}$$

Using the method introduced in [8] we obtain the following equation:

$$\left[i\frac{\partial}{\partial z}+i\beta'(\omega_0)\frac{\partial}{\partial t}-\frac{\beta''(\omega_0)}{2}\frac{\partial^2}{\partial t}+\right.$$
$$\left.+\sum_{p=3}^{\infty}\frac{i^p}{p!}\left(\frac{\partial^p\beta(\omega)}{\partial\omega^p}\right)_{\omega_0}\frac{\partial^p}{\partial t^p}\right]E(z,t)e^{ik\omega_0 t-ik_0 z}+$$
$$+\left[1+i\left(\frac{1}{\omega_0}-\frac{n'(\omega_0)}{n(\omega_0)}\right)\frac{\partial}{\partial t}+\right.$$
$$+\left(\frac{n'(\omega_0)}{n(\omega_0)\omega_0}-\left(\frac{n'(\omega_0)}{n(\omega_0)}\right)^2+\frac{n''(\omega_0)}{2n(\omega_0)}\right)\frac{\partial^2}{\partial t^2}+$$
$$\left.+\sum_{q=3}^{\infty}\frac{i^q}{q!}\frac{\beta(\omega_0)}{(\omega_0/c)^2}\left(\frac{\partial^q}{\partial\omega^q}\frac{(\omega/c)^2}{\beta(\omega)}\right)_{\omega_0}\frac{\partial^q}{\partial t^q}\right]\times \tag{7}$$
$$\times\frac{(\omega_0/c)^2}{2\beta(\omega_0)\varepsilon_0}P_{nl}(z,t)e^{ik\omega_0 t-ik_0 z}+$$
$$+\sum_{q=0}^{\infty}\sum_{m=2}^{\infty}\left\{\frac{i^q(2m-3)!!}{(-1)^{m-1}\varepsilon_0^m q!(2m)!!}\left(\frac{\partial^q}{\partial\omega^q}\frac{(\omega/c)^{2m}}{\beta^{2m-1}(\omega)}\right)_{\omega_0}\times\right.$$
$$\left.\times\frac{\partial^q}{\partial t^q}\left(\varphi^m(z,t)e^{ik\omega_0 t-ik_0 z}\right)\right\}=0$$

The quantities

$$\varphi^m(z,t)=F^{-1}\left\{\frac{P_{nl}^m(k+k_0,\omega+\omega_0)}{E^{m-1}(k+k_0,\omega+\omega_0)}\right\}= \tag{8}$$
$$=F^{-1}\left\{\frac{\left[F\{P_{nl}(z,t)\}\right]^m}{F\{E(z,t)\}^{m-1}}\right\}$$

are higher-order perturbations, F and F^{-1} denote the Fourier and the inverse Fourier Transforms. The notations $\beta'(\omega_0)$; $\beta''(\omega_0)$; $n'(\omega_0)$; $n''(\omega_0)$;… are first-order and second-order derivatives of the respective functions, calculated at the value ω_0. We will further use Eq. (7) to describing different optical phenomena in the subsequent sections.

II.2. Solitons

Equation (7) with the concrete form for the nonlinear polarization (5) and the initial condition for the input pulse permit us to consider the pulse evolution in the propagation within the medium. It is the most general form for the one-dimensional case because it contains all orders of the dispersion and the nonlinearity. This equation is very complicated and finding a general analytic method (given for example in [5]) for this equation is practically "mission impossible", so we should reduce it to a simpler approximate form. It may be worthwhile to look at some simple solutions to the general nonlinear partial differential equation (NPDE) before starting the big machine of any general analytic or numerical scheme for solving it. Without a detailed study of symmetries, we may expect that, among others, our integrable equations will have solutions in the form of a travelling wave. The travelling wave is a solution of the form

$$u(x,t)=U(z),\quad z=x-Vt \tag{9}$$

for equations in which the variable may be interpreted as a real wave function. When the wave consists of a single travelling bump (a displacement from an unperturbed state) or a travelling shock (kink) we call it a solitary wave. This name is extended to the solutions of equations like the nonlinear Schrödinger (NLS), which describe evolution of an envelope of fast oscillations and it is only the envelope of a (usually complex) wave function which has the form of a travelling wave (9). Finally, two or more solitary waves may travel at the same time with different velocities; if they travel towards each other, they would "collide" sooner or later. If they get through the collision unchanged (except for a possible shift in their positions and phases), they are called **solitons**.

For the special case when the medium is isotropic and the medium is of the Kerr type we obtain the cubic nonlinear Schrödinger equation (NLS) which describes the propagation of light pulses in fibers [4, 8]. Optical soliton in fiber exists because of the exact balancing between the group velocity dispersion (GDV) and its counterpart self-phase modulation (SPM). SPM is the nonlinear effect due to the lowest dominant nonlinear susceptibility in silica fibers. One of the most famous physicists working in this domain wrote that the parameters of fiber are a gift from God and that it is a sin not to use solitons in telecommunication! But generally we should take into account the higher order contributions [8, 13, 14].

In some other cases we can also find soliton solutions in an analytic way [5]. In our works on the equations which describe interaction of higher harmonics with the fundamental mode in a laser beam [6, 7] we applied the Hirota scheme to the systems of equations

$$\begin{aligned} &iU_{,z}+U_{,tt}+U^*W=0,\\ &iW_{,z}+PW_{,tt}\pm U^2=0. \end{aligned} \tag{10}$$

for the 2-nd harmonic and

$$\begin{aligned} &iu_z+u_{xx}-u+\left\lfloor(1/9)|u|^2+2|\omega|^2\right\rfloor u+(1/3)u^{*2}\omega=0\\ &i\sigma\omega_z+\omega_{xx}-\alpha u+\left(9|\omega|^2+2|u|^2\right)\omega+(1/9)u^3=0 \end{aligned} \tag{11}$$

for the 3-rd, where the U and u are the amplitudes of the fundamental frequency modes, while the W and w are the amplitudes of the 2-nd and 3-rd harmonics, respectively (all of them rescaled to reduce the number of coefficients). The equations describe propagation of these nonlinearly interacting modes along a waveguide.

We have found that the Hirota scheme worked merely for the exact resonance cases, i.e. not only had the frequencies of the higher harmonics found to be multiplies of the fundamental one, but also the ratio of the dispersion coefficients had to be equal to the ratio of frequencies. Moreover, the only solitary wave solutions of that type were single travelling waves. For the amplitudes of 2-nd harmonic we found a new equation of the NLS type which they satisfy, namely

$$iU_z + U_{tt} \pm \sqrt{2}|U|U = 0. \tag{12}$$

In the case of the ultrashort pulses, with the use of specific properties of their spectrum and intensity, we can simplify the Eq. (7) through neglecting the higher-order nonlinear perturbations and merely preserving the linear and the nonlinear terms with their lower-order derivatives. Before doing this we should consider below in more detail the nonlinear polarization of the medium in the propagation of the ultrashort pulses.

II.3. Raman response function

The nonlinear polarization of the medium is given by (5), where the property of the medium is characterized by the quantity $\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3)$. Apart from its dependencies of the microscopic structure of the molecules and their ordering in the medium, it depends also on the characteristics of the propagating pulses. The microscopic processes have usually the characteristic time of femtoseconds (the characteristic time for the electron response is of the order 0.1 fs, for the nuclei and lattice 10 fs [17]). For the picosecond pulses the nonlinear response of the medium can be considered as instantaneous. In this case the nonlinear susceptibility can be written as follows [2, 3, 17]

$$\begin{aligned} &\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3) = \\ &= \chi^{(3)}\delta(t-t_1)\delta(t-t_2)\delta(t-t_3). \end{aligned} \tag{13}$$

Here $\chi^{(3)}$ is a real constant of the order 10^{-22} m/V^2 and $\delta(t-t_i)$ (i = 1, 2, 3) are the Dirac functions. The reduced equation obtained in this case from (7) is the well-known NLS equation [1, 2, 4, 11]. It describes perfectly the experimental observations for the propagation process.

When input pulses are shorter than 4-5 ps (tens or hundreds fs) the assumption of the instantaneous response of the medium is no longer valid because the time width of the pulses is comparable with the characteristic times of the microscopic processes. Some additional terms describing the delayed response of the medium should be included in the expression (13). This delayed response is related to the reduced Raman scattering on the molecules of the medium [15, 20]. Using the Lorentz atomic model in the adiabatic approximation [1, 15, 17] we can present the nonlinear susceptibility of the medium in the form [3, 17]:

$$\begin{aligned} &\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3) = \\ &= \chi^{(3)}\left[(1-f_R)\delta(t-t_1) + f_R h_R(t-t_1)\right]\delta(t-t_2)\delta(t-t_3). \end{aligned} \tag{14}$$

In the expression for the nonlinear susceptibility (14) we have two contributions, one of the electron layer and one of the nuclei plus the crystal lattice. The electron response is considered as instantaneous, and the delayed response of the nuclei and the lattice is characterized by the function $h_R(t)$ called the **Raman response function**. It has the following form [2, 15, 17]:

$$h_R(t) = \frac{\tau_1^2 + \tau_2^2}{\tau_1\tau_2^2} e^{-t/\tau_2} \sin(t/\tau_1). \tag{15}$$

The Raman response function satisfies the normalization condition $\int_0^\infty h_R(t)\,dt = 1$. The constants f_R, τ_1 and τ_2 depend on the medium. The Fourier Transform of the $h_R(t)$ (called also the Raman response function, but at the frequency ω)) has the following form

$$g_R(\omega) = \frac{1/\tau_1^2 + 1/\tau_2^2}{-\omega^2 - 2i\omega/\tau^2 + \left(1/\tau_1^2 + 1/\tau_2^2\right)}. \tag{16}$$

The imaginary part of $g(\omega)$ is called the Raman amplification function [17, 18, 20].

II.4. Generalized Nonlinear Schrödinger Equation

Substituting the expression (14) into (5), after expanding the terms containing the powers of the intensity of the electric field and neglecting the high-order harmonics (because the phase-matching condition is not fulfilled), we obtain the following expression for the nonlinear polarization:

$$\begin{aligned} P_{nl}(z,t) = \frac{3\varepsilon_0\chi^{(3)}}{8}\Bigg[&(1-f_R)|A(z,t)|^2 A(z,t) + f_R A(z,t)\times \\ &\times \int_{-\infty}^{t} h_R(t-t_1)|A(z,t)|^2\,dt_1 + c.c\Bigg]. \end{aligned} \tag{17}$$

The physical properties of the medium do not depend on the choice of the beginning of the time scale, so the second term in (17) can be rewritten in the form:

$$\int_{-\infty}^{t} h_R(t-t_1)|A(z,t)|^2\,dt_1 = \int_{0}^{\infty} h_R(t_1)|A(z,t-t_1)|^2\,dt_1. \quad (18)$$

Expanding to the first order of the square of the module of the envelope under the integral sign in (18) and using the normalization condition for the function $h_R(t)$ lead to the result

$$\int_{0}^{\infty} h_R(t_1)|A(z,t-t_1)|^2\,dt_1 \approx |A(z,t)|^2 - \frac{T_R}{f_R}\frac{\partial |A(z,t)|^2}{\partial t}, \quad (19)$$

where T is the characteristic time for the Raman scattering effect:

$$T_R = f_R \int_{0}^{\infty} t h_R(t)\,dt. \quad (20)$$

From these results we can write the nonlinear polarization in the form:

$$P_{nl}(z,t) = \\ = \frac{3\varepsilon_0\chi^{(3)}}{8}\left[A(z,t)|A(z,t)|^2 + T_R A(z,t)\frac{\partial |A(z,t)|^2}{\partial t} + c.c\right]. \quad (21)$$

As it has been recognized above, the general Eq. (7) is very complicated, so we should reduce it into an approximate form. It is worth noting that the time and intensity characters of the ultrashort pulses are quite different in comparison to that of the short pulses. It follows that their spectrum is much broader and the pulse power is larger, so in the Eq. (7) we should consider the third-order dispersion terms [2, 3, 11] and the first-order term of the Kerr nonlinearity [1, 4].

Substituting the expression for the nonlinear polarization (21) into (7), after omitting the fast oscillating terms and neglecting the high-order derivatives of the nonlinear term we obtain the following simplest approximate pulse propagation equation:

$$i\frac{\partial A(z,t)}{\partial z} + i\beta'(\omega_0)\frac{\partial A(z,t)}{\partial t} - \frac{\beta''(\omega_0)}{2}\frac{\partial^2 A(z,t)}{\partial t^2} + \\ - \frac{i\beta'''(\omega_0)}{6}\frac{\partial^3 A(z,t)}{\partial t^3} + \\ + \gamma\left[|A(z,t)|^2 A(z,t) + i\tau_s\frac{\partial |A(z,t)|^2 A(z,t)}{\partial t} + \\ - T_R A(z,t)\frac{\partial |A(z,t)|^2}{\partial t}\right] = 0 \quad (22)$$

where

$$\gamma = \frac{3\chi^{(3)}\omega_0}{8n(\omega_0)c}, \quad \tau_s = \frac{1}{\omega_0}\frac{n'(\omega_0)}{n(\omega_0)} \approx \frac{1}{\omega_0}. \quad (23)$$

)

Using the new parameters and variables

$$L_D = \frac{\tau_0^2}{|\beta''(\omega_0)|}, \quad L_{NL} = \frac{1}{\gamma P_0}, \quad N^2 = \frac{L_D}{L_{NL}},$$

$$\delta_3 = \frac{\beta'''(\omega_0)}{6|\beta''(\omega_0)|\tau_0}, \quad S = \frac{\tau_s}{\tau_0}, \quad \tau_R = \frac{T_R}{\tau_0}, \quad (24)$$

$$\tau = \frac{t-\beta'(\omega_0)z}{\tau_0}, \quad U(\xi,\tau) = \frac{1}{\sqrt{P_0}}A(z,t), \quad \xi = \frac{z}{L_D},$$

where τ_0 and P_0 stand respectively for the time width and the maximal power at the top of the envelope function, we can rewrite the Eq. (22) in the normalized form:

$$\frac{\partial U}{\partial \xi} = -\mathrm{sign}(\beta''(\omega_0))\frac{i}{2}\frac{\partial^2 U}{\partial \tau^2} + \delta_3\frac{\partial^3 U}{\partial \tau^3} + \\ + iN^2\left(|U|^2 U + iS\frac{\partial}{\partial \tau}\left(|U|^2 U\right) - \tau_R U\frac{\partial |U|^2}{\partial \tau}\right). \quad (25)$$

The Equation (25) is the lowest-order approximate form when we consider the higher-order dispersion and nonlinearity effects in the general propagation Eq. (7). It is one of the most useful approximate forms describing the propagation process of the ultrashort pulses, called the generalized nonlinear Schrödinger equation (GNLS) [3, 11, 16, 17]. It has a more complicated form than the nonlinear Schroedinger equation describing the propagation of the short pulses [1, 2, 4, 11] as it contains the higher-order dispersive and nonlinear terms. The parameters characterizing these effects: δ_3, S, τ_R govern respectively the effects of TOD, self-steepening and the self-shift frequency. In the formulas (24) we see that when τ_0 decreases, i.e. the pulse is shorter and the magnitude of these parameters increases, the higher-order effects should be considered (see Subsection II.5).

Under the influence of TOD both the pulse shape and the spectrum change in a complicated way. When the propagation distance is larger the oscillation of the envelope function is stronger, creating a long trailing edge to the later time, and the spectrum is broadened to two sides and splits into several peaks [2, 11].

Self-steepening of the pulse leads to the formation of a steep front in the trailing edge of the pulse, resembling the usual shock wave formation. This effect is called the optical shock. The pulse becomes more asymmetric in the propagation and its tail finally breaks up [1, 4, 11, 16].

In the stimulated Raman scattering the Stokes process is more effective than the anti-Stokes process [2, 20]. This fact leads to the so-called self-shift frequency of the pulse. As a result the spectrum is shifted down to the low-frequency region. In other words, the medium "amplifies" the long wavelength parts of the pulse. The pulse loses its energy and undergoes a complex change when it enters deeply into medium.

For the ultrashort pulses with the width $\tau_0 \approx 50\text{ fs}$ and the carrier wavelength $\lambda_0 = 1.55\ \mu\text{m}$, the higher-order parameters in (24) during their propagation in the medium SiO_2 have the values of $\delta_3 \approx 0.03$, $S \approx 0.03$, $\tau_R \approx 0.1$. These values are smaller than one, so the higher-order effects are considered as the perturbations in comparison with the Kerr effect. Therefore, when the pulse propagates in a silica optical fiber, the self-shift frequency effect dominates over the TOD and the self-steepening for the pulses with the width of hundreds and tens femtoseconds. The self-steepening becomes important only for the pulses of nearly 3 fs [2, 11].

When t has the value of picoseconds or larger, the values of δ_3, S and τ_R are very small and they can be neglected. Equation (25) reduces to the well-known NLS equation for the short pulses [1, 2, 4]. As it has been recognized above, NLS can be solved by the Inverse Scattering Method [5], but this Method cannot be applied to the Eq. (25) any more. The problem of finding a general analytic method for this equation is practically a "mission impossible" except some special cases, when some specific conditions should be satisfied. A review of some analytic methods as the Inverse Scattering Method, Hirota's Method and Painleve's Test is given in [5]. We describe below two very useful methods which are not considered there, namely the *developed Jacobi elliptic function expansion* and the *variational method*. Presentation of several numerical methods of finding approximate solutions of the Equation (25) is the subject of Section III.

II.5. Developed Jacobi elliptic function expansion

Following [12] we consider a nonlinear partial differential equation in a general form

$$N\left(F, |F|, \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial t^2}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial x \partial t} \cdots\right) = 0\,. \tag{26}$$

We seek the traveling wave solutions of the form

$$F = u(\xi)\, e^{-(kx-\omega t)}, \quad \xi = cx - \lambda t + x_0, \tag{27}$$

where $u(\xi)$ is a real function, λ is a constant parameter and k and ω denote the wave number and the frequency, respectively. Substituting (27) into (26) we obtain an ordinary differential equation

$$N\left(u, \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \frac{d^3u}{d\xi^3}, \cdots\right) = 0. \tag{28}$$

We take the *ansatz* of the solution in the form of a finite series of Jacobi elliptic functions $cn(\xi,m)$ (or $sn(\xi,m)$), i.e.

$$u(\xi) = \sum_{i=0}^{n} a_j cn^j(\xi). \tag{29}$$

Here a_j are constants which will be determined later, and the highest degree of the function u is

$$O\big(u(\xi)\big) = n. \tag{30}$$

It follows from properties of Jacobi elliptic functions that the highest degree of derivatives is taken as

$$O\big(d^p\, u(\xi)/d\xi^p\big) = n + p. \tag{31}$$

n in (29) is selected in such a way that the highest degree of derivatives is equal to the degree of the nonlinear term. Substituting (29) into (28) and equating the coefficients of all power of $cn(\xi)$. $sn(\xi)$, $dn(\xi)$ to zero leads to a set of algebraic equations for a_j. By solving these equations, we obtain the final result for u in the form (29). We will apply this method below to find the soliton solutions in two cases: for the Eq. (12) and GNSE.

II.5.1. Equation (12) for the second harmonic

We look for the soliton solutions for the Eq. (12) introduced above:

$$iU_z + U_{tt} \pm \sqrt{2}\,|U|U = 0. \tag{32}$$

Performing the transformation

$$U = V(\xi)\exp\big[t(kz - \omega t)\big]; \qquad \xi = ct - \lambda z + z_0 \tag{33}$$

we have

$$\begin{aligned}
\frac{\partial U}{\partial z} &= \left[\frac{\partial V}{\partial z} + ikV\right] e^{i(kz-\omega t)} = (-\lambda V' + ikV)\, e^{i(kz-\omega t)}, \\
\frac{\partial U}{\partial t} &= \left[\frac{\partial V}{\partial z} - i\omega V\right] e^{i(kz-\omega t)} = (cV' - i\omega V)\, e^{i(kz-\omega t)}, \\
\frac{\partial^2 U}{\partial t^2} &= \left[c^2 V'' - 2i\omega c V' - \omega^2 V\right] e^{i(kz-\omega t)}, \\
|U|U &= V^2 e^{i(kz-\omega t)},
\end{aligned} \tag{34}$$

where for the sake of simplicity we assume $|V| = V$. Substituting the expressions (34) to Eq. (32) leads to the system of equations

$$\begin{cases} \lambda = 2\omega c \\ c^2 V'' + \sqrt{2} V^2 - \left(\omega^2 + k\right) V = 0. \end{cases} \tag{35}$$

We look for the solution of (35) in the form

$$V = a_0 + a_1 cn(\xi) + a_2 cn^2(\xi). \tag{36}$$

Solving (35) for the case $m = 1$ we obtain

$$a_2 = 3c^2\sqrt{2};\quad a_1 = 0;\quad a_0 = -2\sqrt{2}c^2;$$

$$c = c,\ \omega = \omega;\quad k = -4c^2 - \omega^2;$$

$$V = -2\sqrt{2}c + 3c^2\sqrt{2}\sec h^2\left(ct - kz + z_0\right).$$

We draw below in Figs. 1 and 2 the solutions (36) for the values c = ″ and c = L′, respectively.

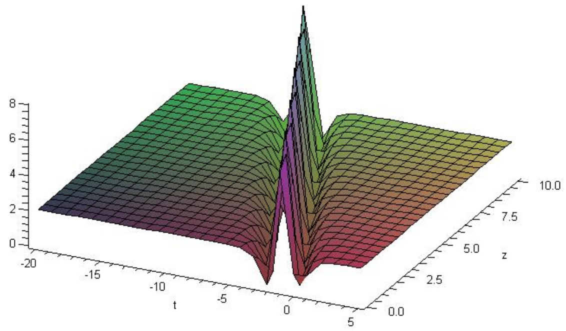

Fig. 1. Solution (36) with $c = 1/2$

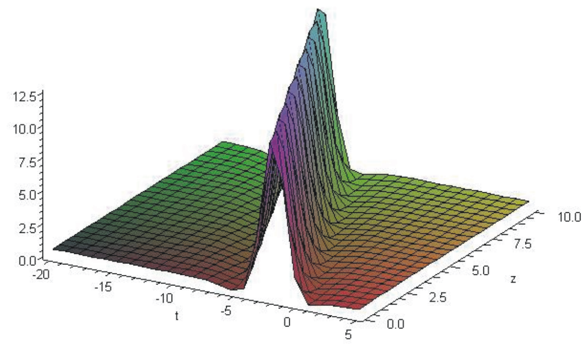

Fig. 2. Solution (36) with $c = 1/4$

II.5.2. GNLS with the four-order dispersion

As it has been emphasized above, in the case of ultrashort light pulses (femtosecond pulses which have much potential for future technology), in comparison with the nonlinear Schrödinger equation (NLS), higher-order terms should be taken into account. For this reason, we consider the GNLS in the form

$$\begin{aligned} \frac{\partial E}{\partial z} &= i\left(\alpha_1 \frac{\partial^2 E}{\partial t^2} + \alpha_2 \frac{\partial^4 E}{\partial t^4} + \alpha_3 |E|^2 E\right) + \alpha_4 \frac{\partial^3 E}{\partial t^3} + \\ &+\ \alpha_5 \frac{\partial\left(|E|^2 E\right)}{\partial t^3} + \alpha_6 E \frac{\partial |E|^2}{\partial t}, \end{aligned} \tag{37}$$

where the real parameters $\alpha_i\ (i = 1, \ldots, 6)$ have the following physical interpretations: α_1 corresponds to the group velocity dispersion (GVD), α_2 to the four-order dispersion (FOD), α_3 to self-phase modulation (SPM), α_4 to third-order dispersion (TOD), α_5 to self-steepening (SS) and α_6 to the self frequency shift (SFS) arising from stimulated Raman scattering (SRS). Thus in comparison with Eq. (25), the FOD is included. In order to find traveling wave solutions of Eq. (37), we use the developed Jacobi elliptic function expansion method described above. Firstly, we look for the electric field in the form

$$E(z,t) = u(\xi)\exp\left[i(kz - \omega t)\right],\qquad \xi = ct - \lambda z + z_0. \tag{38}$$

Substituting (38) into (37) we obtain

$$\begin{aligned} &-\lambda u' + iku = \\ &= i\alpha_2\left[c^4 u'''' + \omega^4 u - 6c^2\omega^2 u'' + 4i\left(c\omega^3 u' - c^3\omega u'''\right)\right] + \\ &+\ i\alpha_1\left(c^2 u'' - 2ic\omega u' - \omega^2 u\right) i\alpha_3 u^3 + \alpha_5\left(3cu^2 u' - i\omega u^3\right) + \\ &+\ \alpha_4\left[c^3 u''' - 3c\omega^2 u' + i\left(\omega^3 u - 3c^2\omega u''\right)\right] + 2\alpha_6 c u^2 u'. \end{aligned} \tag{39}$$

Separating the real and imaginary parts of this equation leads to the following system of equations

$$\begin{aligned} &c_3\left(\alpha_4 + 4\alpha_2\omega\right)u''' + \left(\lambda - 3a_4 c\omega^2 + 2\alpha_1 c\omega - 4\alpha_2 c\omega^3\right)u' + \\ &+\left(3\alpha_5 + 2\alpha_6\right)cu^2 u' = 0, \end{aligned} \tag{40a}$$

$$\begin{aligned} &c^4\alpha_2 u'''' + \left(c^2\alpha_1 - 6\alpha_2 c^2\omega^2 - 3c^2\omega\alpha_4\right)u'' + \\ &+\left(-k - \alpha_1\omega^2 + \alpha_2\omega^4 + \alpha_4\omega^3\right)u + \left(\alpha_3 - \alpha_5\omega\right)u^3 = 0. \end{aligned} \tag{40b}$$

Now we calculate the highest degree of derivative $O\left(d^4 u(\xi)/d\xi^4\right) = n + 4$ and the degree of the nonlinear term $O\left(u^3(\xi)\right) = 3n$. Equating these numbers leads to $n = 2$. It follows that the function $u(\xi)$ can be found from the form

$$u(\xi) = a_0 + a_1 cn(\xi) + a_2 cn^2(\xi). \tag{41}$$

For the sake of simplicity, we suppose that $a_0 = a_1 = 0$, then $u(\xi) = a_2 cn^2(\xi)$. Substituting this expression into Eq. (40b) gives

$$A_6 cn^6(\xi) + A_4 cn^4(\xi) + A_2 cn^2(\xi) + A_0 = 0, \tag{42}$$

where coefficients A_i contain different parameters involved in the problem. Equating the coefficient of the first term in 42 to zero leads to $A_6 = 48\alpha_2 c^4 a_2 m^4 + 24\alpha_2 c^4 a_2 m^2 = 0$. Because a_2, m, c should be different from zero, we have $a_2 = 0$. This means that if the term FOD is taken into account, the traveling wave solutions do not exist. We conclude that for the existence of solutions in this type, the orders of dispersion higher than three should not be taken into account. Then we can rewrite (40a) and (40b) in the form (with $\alpha_2 = 0$)

$$\begin{aligned} &c^3\alpha_4 u''' + \left(\lambda - 3\alpha_4 c\omega^2 + 2\alpha_1 c\omega\right)u' + \\ &+ \left(3\alpha_5 + 2\alpha_6\right)cu^2 u' = 0, \end{aligned} \tag{43a}$$

$$\begin{aligned} &\left(c^2\alpha_1 - 3c^2\omega\alpha_4\right)u'' + \left(-k - \alpha_1\omega^2 + \alpha_4\omega^3\right)u + \\ &+ \left(\alpha_3 - \alpha_5\omega\right)u^3 = 0. \end{aligned} \tag{43b}$$

Differentiating two sides of Eq. (43b) with respect to the ξ gives us

$$\begin{aligned} &u''' + \frac{\left(-k - \alpha_1\omega^2 + \alpha_4\omega^3\right)}{\left(c^2\alpha_1 - 3c^2\omega\alpha^4\right)}u' + \\ &+ 3\frac{\left(\alpha_3 - \alpha_5\omega\right)}{\left(c^2\alpha_1 - 3c^2\omega\alpha_4\right)}u^2 u' = 0. \end{aligned} \tag{44}$$

Comparing (44) with (43a) leads to formulas for ω and k:

$$\omega = \left[\alpha_1\left(3\alpha_5 + 2\alpha_6\right) - 3\alpha_3\alpha_4\right]\left[6\alpha_4\left(\alpha_5 + \alpha_6\right)\right]^{-1}, \tag{45}$$

$$k = -\frac{1}{c\alpha_4}\left(\lambda - 3\alpha_4 c\omega^2 + \alpha_1 c\omega\right) - \alpha_1\omega^2 + \omega^3\alpha_4. \tag{46}$$

Then Eqs. (43a), (43b) reduce to

$$u'' + Au + Bu^3 = 0, \tag{47}$$

where

$$A = \frac{2\alpha_1\omega + \lambda - 3\alpha_4\omega^2}{c^2\alpha_4}; \qquad B = \frac{3\alpha_5 + 2\alpha_6}{3c^2\alpha_4}. \tag{48}$$

Now, we use the formalism described above for Eq. (47). Firstly, we calculate $O\left(d^2u/d\xi^2\right) = n + 2$ and $O\left(u^3(\xi)\right) = 3n$. Then $n = 1$ and we can write $u(\xi)$ in the following form:

$$u(\xi) = a_0 + a_1 cn(\xi). \tag{49}$$

Substituting (49) into (47) and equating the coefficients of all powers of $cn(\xi)$ to zero yields the values of unknown parameters a_0, a_1, c, λ. We have performed this step by MAPLE and obtained:

$$\begin{aligned} &a_0 = 0, \quad a_1 = \sqrt{\frac{6\alpha_4}{2\alpha_6 + 3\alpha_5}}mc, \\ &\lambda = -2m^2c^2\alpha_4 - 2\alpha_1\omega + c^2\alpha_4 + 3\alpha_4\omega^2, \end{aligned} \tag{50}$$

while c is an arbitrary constant and m is the modulus number of the Jacobi elliptic functions. Then the traveling wave solutions of the propagation Eq. (37) have the following form

$$\begin{aligned} &E(z,t) = \sqrt{\frac{6\alpha_4}{2\alpha_6 + 3\alpha_5}}mc.\exp\left[i\left(kz - \omega t\right)\right] \\ &cn\left[ct - \left(-2m^2c^2\alpha_4 - 2\alpha_1\omega + c^2\alpha_4 + 3\alpha_4\omega^2\right)z + z_0\right], \end{aligned} \tag{51}$$

where the expressions for k, ω are given by (45) and (46). When m tends to 1, we obtain a bright soliton solution

$$\begin{aligned} &E(z,t) = \sqrt{\frac{6\alpha_4}{2\alpha_6 + 3\alpha_5}}c\exp\left[i\left(kz - \omega t\right)\right] \\ &\operatorname{sech}\left[ct - \left(-c^2\alpha_4 - 2\alpha_1\omega + 3\alpha_4\omega^2\right)z + z_0\right]. \end{aligned} \tag{52}$$

Now instead of (49) we use the *ansatz*

$$u(\xi) = a_0 + a_1 sn(\xi). \tag{53}$$

By substituting (53) into (47) we obtain

$$\begin{aligned} &a_0 = 0, \quad a_1 = \sqrt{\frac{-6\alpha_4}{2\alpha_6 + 3\alpha_5}}mc, \\ &\lambda = c^2\alpha_4 - 2\alpha_1\omega + m^2c^2\alpha_4 + 3\alpha_4\omega^2, \end{aligned} \tag{54}$$

while c is also an arbitrary constant. Then the solution of Eq. (37) has the following form:

$$\begin{aligned} &E(z,t) = \sqrt{\frac{-6\alpha_4}{2\alpha_6 + 3\alpha_5}}mc.\exp\left[i\left(kz - \omega t\right)\right] \\ &sn\left[ct - \left(c^2\alpha_4 - 2\alpha_1\omega + m^2c^2\alpha_4 + 3\alpha_4\omega^2\right)z + z_0\right], \end{aligned} \tag{55}$$

where the expressions for k, ω are given by (45) and (46). When the modulus number m tends to 1 we have a dark soliton solution in the following form:

$$\begin{aligned} &E(z,t) = \sqrt{\frac{-6\alpha_4}{2\alpha_6 + 3\alpha_5}}c \\ &\tanh\left[ct - \left(2c^2\alpha_4 - 2\alpha_1\omega + 3\alpha_4\omega^2\right)z + z_0\right]\exp\left[i\left(kz - \omega t\right)\right]. \end{aligned} \tag{56}$$

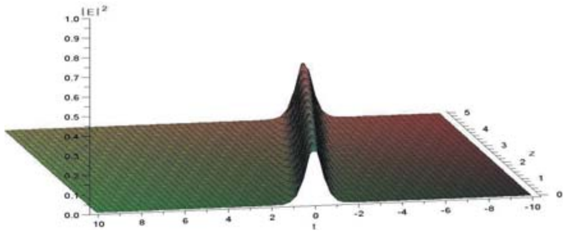

Fig. 3. Bright soliton (52) with $c = 2$, $\alpha_1 = -0.5$, $\alpha_3 = 1$, $\alpha_4 = 1/24$, $\alpha_5 = 0.8$, $\alpha_6 = 0.5$ and $z_0 = 0$

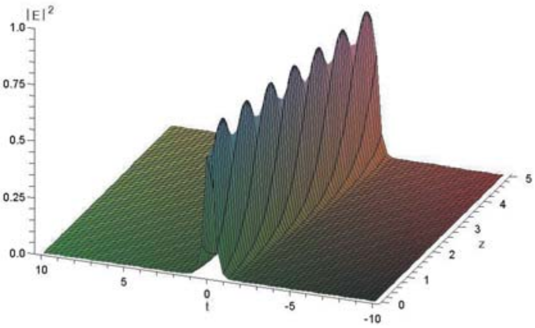

Fig. 4. Bright soliton (52) with $c = 3$, $\alpha_1 = -0.5$, $\alpha_3 = 1$, $\alpha_4 = 1/24$, $\alpha_5 = 0.8$, $\alpha_6 = 0.5$ and $z_0 = 0$

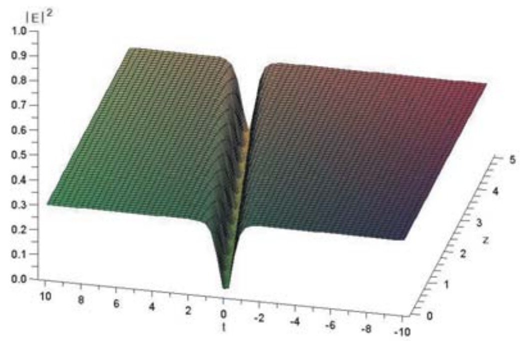

Fig. 5. Dark soliton (55) with $c = 2$, $\alpha_1 = -0.5$, $\alpha_3 = 1$, $\alpha_4 = -1/24$, $\alpha_5 = 0.8$, $\alpha_6 = 0.5$ and $z_0 = 0$

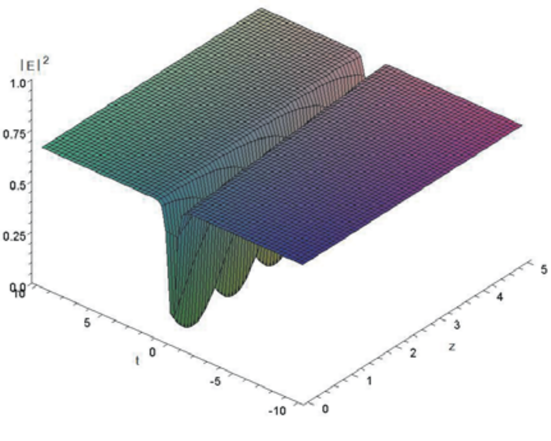

Fig. 6. Dark soliton (55) with $c = 3$, $\alpha_1 = 0.5$, $\alpha_3 = 1$, $\alpha_4 = -1/24$, $\alpha_5 = 0.8$, $\alpha_6 = 0.5$ and $z_0 = 0$

Our expressions (52) and (56) are just the results previously obtained by several authors (*e.g.* the formulas (9), (12) in [12] and (56), (58) in [31]). We demonstrate these for some values of parameters involved in Figures 3-6.

II.6. Variational Method

The Variational Method (VM) is a powerful tool in finding multidimensional soliton solutions of nonlinear optical systems or Bose-Einstein condensations. In the some problems considered here, we present the good agreement between predictions of variational method and direct numerical calculations. It is interesting to note that the estimate obtained from a simple variational model can be in good agreement with numerical results even for complicated systems.

The main idea of the variational method is to replace the nonlinear partial differential equation (we illustrate this following the example of the Nonlinear Schrödinger equation – NLS which is derived above and becomes one of the basic equations of modern mathematical physics) with a system of ordinary differential equations (ODEs) – which are much easier in considerations. In the case of multidimensional system the original NLS is even very hard to solve numerically, so that the VM is used for giving us brief information about the system and a good hint for performing numerical confirmation. The VM used for finding optical solitons has been initiated by the papers of Anderson *et al.* [32] and reviewed perfectly by Malomed [33]. Within Lagrangian formalism, the evolution equation is derived from minimal principle of the *action functional S*:

$$S = \int L dt \tag{57}$$

where

$$L = \int \Im \, dxdy. \tag{58}$$

L is known as Lagrangian while $\Im$ is called *Lagrangian density*, t is evolution variable and x, y are spatial variables. The Lagrangian density is functional of the wave-function (or the slowly varying amplitude in nonlinear optics) of the system, its partial derivatives with respect to above variables and their corresponding complex conjugates:

$$\Im = \Im\left(\psi, \psi^*, \psi_t, \psi_t^*, \psi_x, \psi_x^* \psi_y, \psi_y^*\right). \tag{59}$$

The condition of extremum of the action $\delta S/\delta\psi^* = 0$ leads to Euler-Lagrange equation of the form:

$$\frac{\partial}{\partial t}\frac{\partial \Im}{\partial \psi_t^*} + \frac{\partial}{\partial x}\frac{\partial \Im}{\partial \psi_x^*} + \frac{\partial}{\partial y}\frac{\partial \Im}{\partial \psi_y^*} - \frac{\partial \Im}{\partial \psi^*} = 0. \tag{60}$$

Tis equation is the evolution equation of our system (NLS).

The VM is applied when we manipulate directly the Lagrangian L instead of solving Eq. (60). The variational calculation begins with postulating a trial function (*ansatz*). The ansatz contains a set of variational parameters $X_i(t)$ that are functions of the evolution variable t.

The perspective ansatz is substituted into the Lagrangian density, and the effective Lagrangian $L_{\text{eff}}\left(X_i\left(t\right)\right)$ is obtained by doing integration. Substituting the effective Lagrangian into Equation of the type (60) we obtain finally a set of ODEs:

$$\frac{d}{dt}\frac{\partial L_{\text{eff}}}{\partial X_{it}}-\frac{\partial L_{\text{eff}}}{\partial X_i}=0. \tag{61}$$

Therefore, the VM reduces complex dynamics described by NPDE to a relatively simple system of ODEs governing evolution of the variational parameters.

VA provides a convenient framework to study stationary solutions that correspond to the *fixed points* of the Eq. (61). The fixed points can be found by setting $dX_i/dt=0$ and reducing the ODEs to a system of algebraic equations. Stability of fixed points against small pertubations that can be determined by linearization of Eq. (61) around the fixed points, provides an indication of the stability of the corresponding stationary solutions.

We illustrate this general formalism by showing variational calculations for one dimensional NLS equation which plays the role of a background for another calculations in several papers of nonlinear optics.

The NLS equation reads (this equation is written in “non-optical” notation with the interchange between temporal and spatial variables, we consider attractive interaction, so that $g_{1D} > 0$):

$$i\psi_t=-\frac{1}{2}\psi_{yy}-g_{1D}\left|\psi\right|^2\psi, \tag{62}$$

where we assume that norm of the wavefunction is equal to

$$\int_{-\infty}^{\infty}\left|\psi\right|^2 dy=N. \tag{63}$$

By doing proper rescaling one can easily prove that, in this case, there is only one governed parameter $\ddot{e}_{1D} = g_{1D}N$.

The Lagrangian density has the following form:

$$\Im=\frac{1}{2}\left[i\left(\psi_t\psi^*-\psi_t^*\psi\right)-\left|\psi_y\right|^2+g_{1D}\left|\psi\right|^4\right]. \tag{64}$$

We introduce an *ansatz* of the Gaussian form

$$\psi\left(y,t\right)=A\left(t\right)\exp\left[-\frac{y^2}{2}\left(\frac{1}{V\left(t\right)^2}-ib\left(t\right)\right)+i\varphi\left(t\right)\right], \tag{65}$$

with variational parameters: amplitude $A(t)$, chirp $b(t)$, overall phase $\varphi(t)$ and width $V(t)$. By substituting *ansatz* (65) into our Lagrangian density and integrating over y we obtain the effective Lagrangian

$$L=\sqrt{\pi}A^2\left[-\varphi V-\frac{\dot{b}V^3}{4}-\frac{1}{4V}-\frac{b^2V^3}{4}+\frac{g_{1D}A^2V}{2\sqrt{2}}\right]. \tag{66}$$

It leads to the Euler-Lagrange equations of variational parameters:

$$\begin{gathered}\dot{\varphi}=\frac{g_{1D}A^2}{\sqrt{2}}-\frac{1}{4V^2}-\frac{V\ddot{V}}{4},\\ \dot{V}=\frac{1}{V^3}-\frac{g_{1D}}{\sqrt{2\pi}V^2},\\ b=\frac{\dot{V}}{V}.\end{gathered} \tag{67}$$

The Equation (62) conserves the total norm, therefore we have additional equation which interprets this fact:

$$A^2V=\text{const}, \tag{68}$$

and which is related to the norm of the ansatz

$$\int_{-\infty}^{\infty}\left|\psi\right|^2 dy=A^2V\sqrt{\pi}=N\Rightarrow A=\sqrt{\frac{N}{V\sqrt{\pi}}}. \tag{69}$$

In the case of stationary solutions, we set $\ddot{V}=\dot{V}=0$. These conditions can be satisfied when

$$V=\frac{\sqrt{2\pi}}{g_{1D}N} \tag{70}$$

and $b = 0$.

Here we should notice that for the stationary solutions one can write:

$$\psi\left(y,t\right)=\varphi\left(y\right)\exp\left(-i\mu_{1D}t\right), \tag{71}$$

where μ_{1D} stands for eigenvalue of the solution. In nonlinear optics this eigenvalue corresponds to the wavevector of soliton and in the theory of Bose-Einstein condensates it becomes chemical potential. Here the soliton is a bound state so that its eigenvalue should be negative. From the Euler Lagrange Eq. (67) we can immediately evaluate such eigenvalue. By substituting V into the equation for phase we obtain

$$\dot{\varphi} = -\mu_{1D} = \frac{3g_{1D}^2 N^2}{8\pi}. \tag{72}$$

Finally, within VM the soliton solution reads:

$$\psi(y,t) = \frac{N\sqrt{g_{1D}}}{\pi\sqrt{2}} \exp\left(-\frac{N^2 g_{1D}^2}{4\pi} y^2 + i\frac{3g_{1D}^2 N^2}{8\pi} t \right). \tag{73}$$

As it has been emphasized above, the one dimensional NLS equation is fully integrable. It can be solved exactly by means of the inverse scattering technique [5]. The single soliton solution has the form:

$$\psi(y,t) = \frac{N\sqrt{g_{1D}}}{2} \operatorname{sech}\left(\frac{Ng_{1D}}{2} y \right) \exp\left(+i\frac{g_{1D}^2 N^2}{8} t \right). \tag{74}$$

Obviously, the Euler-Lagrange Equations (67) enable us to investigate dynamics of the system if the initial values of variational parameters $A(0)$, $b(0)$, $\varphi(0)$ and $V(0)$ are given. The VM which is using Lagrangian density (64) and ansatz (65) is known as dynamical approach. It can be applied to more complicated processes, for example, soliton collisions (for example in [33], Chapter IV) .

If we are only interested in the static case, it means that we search for stationary solutions (71) and their stability, then instead of using Lagrangian density (64) and *ansatz* (65) one can adopt simpler Lagrangian density

$$\Im = \mu_{1D} |\psi|^2 - \frac{1}{2}|\psi_y|^2 + \frac{1}{2} g_{1D} |\psi|^4 \tag{75}$$

and *ansatz*:

$$\psi(y,t) = A\exp\left(-\frac{y^2}{2V^2} - i\mu_{1D} t \right). \tag{76}$$

Strictly speaking, the functional (75) is not full Lagrangian density but a potential energy. Here the kinetic term is dropped out because we are not interested in dynamics. The chemical potential μ_{1D} plays a role of Lagrange multiplier. Based on the minimal principle of action one can easily proof that the functional (75) leads to the stationary NLS equation:

$$\mu_{1D}\psi = -\frac{1}{2}\psi_{yy} - g_{1D} |\psi|^2 \psi. \tag{77}$$

The VM based on the functional (75) and *ansatz* (76) is called the stationary approach.

The effective Lagrangian takes the form:

$$L = \sqrt{\pi} A^2 V \left[\mu_{1D} - \frac{1}{4V^2} + \frac{g_{1D} A^2}{2\sqrt{2}} \right]. \tag{78}$$

The Euler-Lagrange equations of variational parameters A and V read:

$$\begin{aligned} &2AV\left[\mu_{1D} - \frac{1}{4V^2} \right] + \frac{4g_{1D} A^3 V}{2\sqrt{2}} = 0, \\ &A^2 \mu_{1D} + \frac{A^2}{4V^2} + \frac{g_{1D} A^4}{2\sqrt{2}} = 0. \end{aligned} \tag{79}$$

The relation between norm (63) and parameter A in (76) is also similar to (69). From such equations we immediately reproduce the above results:

$$\begin{aligned} V &= \frac{\sqrt{2\pi}}{g_{1D} N}, \\ \mu_{1D} &= -\frac{3g_{1D}^2 N^2}{8\pi}. \end{aligned} \tag{80}$$

Instead of using Lagrangian, we can apply similar procedures of variational calculations for Hamiltonian. Within the stationary approach, the Hamiltonian density reads:

$$\mathrm{H} = \frac{1}{2}|\psi_y|^2 - \frac{1}{2} g_{1D} |\psi|^4 . \tag{81}$$

One can easily prove that this Hamiltonian density describes the same equation of motion as (77). The adopted ansatz for this case concerning the relation (63) has the following form:

$$\psi(y,t) = \sqrt{\frac{N}{V\sqrt{\pi}}} \exp\left(-\frac{y^2}{2V^2} - i\mu_{1D} t \right). \tag{82}$$

After performing integration over y we obtain effective Hamiltonian:

$$\mathrm{H} = \frac{N}{4V^2} - \frac{g_{1D} N^2}{2V\sqrt{2\pi}}. \tag{83}$$

The Euler-Lagrange equation of parameter V takes the form:

$$\frac{\partial \mathrm{H}}{\partial V} = \frac{g_{1D} N^2}{2V^2\sqrt{2\pi}} - \frac{N}{2V^3}. \tag{84}$$

This equation has solution

$$V = \frac{\sqrt{2\pi}}{g_{1D} N} \tag{85}$$

which is identical to (70). The value of μ_{1D} in the *ansatz* (82) can be evaluated by doing the following calculation:

$$\mu_{1D} = -\frac{\int \left(\frac{1}{2}\psi^* \psi_{yy} + g_{1D} |\psi|^4 \right) dy}{\int |\psi|^2 \, dy} = -\frac{3g_{1D}^2 N^2}{8\pi}. \tag{86}$$

The conclusion of stability is based on checking second derivative of the functional (83) with respect to V at stationary point (85):

$$\frac{\partial^2 \mathrm{H}}{\partial V^2} = \frac{3N}{2V^4} - \frac{g_{1D}^2 N^2}{\sqrt{2\pi}\mathrm{V}^3} = \frac{g_{1D}^4 N^5}{8\pi^2} > 0. \tag{87}$$

This condition corresponds to the case of the stable solution. At this point the Hamiltonian has a local minimum.

In Sec. IV we will apply the formalism of VM described here to consider the collapse of optical solitons.

III. NUMERICAL METHODS TO SOLVE THE PULSE PROPAGATION EQUATION

III.1. Split-Step Algorithm of second order

Firstly we present Split-Step Algorithm for finding approximate solutions of the pulse propagation equation. Equation (25) can be written in the form

$$\frac{\partial U}{\partial \xi} = \left(\hat{L} + \hat{N}(U)\right)U\ , \tag{88}$$

where $\hat{L}$ and $\hat{N}$ are the linear and nonlinear operator respectively acting on the envelope function.

By some calculations performed in [9] we obtain the following formula describing Split-Step algorithm for the problem (88):

$$\begin{gathered} U(\xi + \Delta\xi, \tau) \approx \\ \approx \exp\left(\frac{\Delta\xi}{2}\hat{L}\right)\exp\left(\Delta\xi \hat{N}\left(U(\xi,\tau)\right)\right)\exp\left(\frac{\Delta\xi}{2}\hat{L}\right)U(\xi,\tau). \end{gathered} \tag{89}$$

This expression permits us to specify the approximate value of the envelope function in the location $\xi + \Delta\xi$ from its value in the ξ.

For calculating the value of the envelope function by (29) we should know how the action of the linear and nonlinear operators on the envelope function is calculated. Because these operators contain the time partial derivatives one can calculate them just by Fourier Transform.

We take the value of the time variable in the finite interval [*a, b*] which is so large that its borders do not have any influence on the final results of the calculations. We assume now the periodic condition on borders that $U(\xi,a) = U(\xi,b)$ for $\xi \in [0, \xi_0]$. For convenience, we change the variable in (29) in such a way that it normalizes the interval [*a, b*] into the interval [0, 2π] and we divide this interval into N points with distance between them $\Delta\tau = 2\pi / N$. We denote these points as $\tau_j = 2\pi j / N$, $j = 0, 1, 2, ..., N$. Then we have the Discrete Fourier Transform of the series $U(\xi, \tau - j)$ as follows:

$$\begin{gathered} U(\xi, \omega_k) = F_k\left[U(\xi, \tau_j)\right] = \\ = \frac{1}{N}\sum_{j=0}^{N-1} U(\xi, \tau_j)\exp(-i\omega_k \tau_j), \quad -\frac{N}{2} \le \omega_k \le \frac{N}{2} - 1. \end{gathered} \tag{90}$$

The Inverse Fourier Transform is defined as follows:

$$\begin{gathered} U(\xi, \tau_j) = F_j^{-1}\left[U(\xi, \omega_k)\right] = \\ = \sum_{k=-N/2}^{N/2-1} U(\xi, \omega_k)\exp(i\omega_k \tau_j), j = 0, 1, 2, ..., N. \end{gathered} \tag{91}$$

F here denotes Fourier Transform and F^{-1} denotes its inverse transform. Calculations in (90) and (91) are made effective by the fast algorithm FFT [25]. The time partial derivatives of the envelope function in both the linear and nonlinear operator (27) and (28) can be easily calculated by multiplying the Fourier coefficients $U(\xi, \omega_k)$ by powers of $-i\omega_k$ corresponding to the order of derivative and then taking the Inverse Fourier Transform. For example, second-order derivative of the envelope function in the point (ξ, τ_j) can be calculated as $F_j^{-1}\left[-\omega_k^2 F_k\left[U(\xi, \tau_j)\right]\right]$.

III.2. The fourth order Runge-Kutta algorithm

Equation (25) can be also solved by using the Runge-Kutta algorithm. In this method the time discretization and calculations of time partial derivatives are the same as in the previous subsection, but the spatial derivatives are calculated by Runge-Kutta algorithm. What is applied here is the fourth order Runge-Kutta algorithm, very popular for solving the differential equations [16, 21, 25, 26].

After using Fourier Transform for calculating the time partial derivatives as above, Equation (25) becomes

$$\begin{gathered} \frac{d}{d\xi}\left(F[U]\right) = \left((-i\omega)^2\frac{i}{2} + (-i\omega)^3\delta_3\right)F[U] + \\ + \ iN^2\left[\left(1 + iS(-i\omega)\right)F\left[|U|^2 U\right] + \right. \\ \left. - \ \tau_R F\left[UF^{-1}\left[(-i\omega)F\left[|U|^2\right]\right]\right]\right] \end{gathered} \tag{92}$$

Denoting

$$V = \exp\left(\left(\frac{i\omega^2}{2} - i\omega^3\delta_3\right)\xi\right)F[U], \tag{93}$$

and after some calculations [9] we obtain the value of the

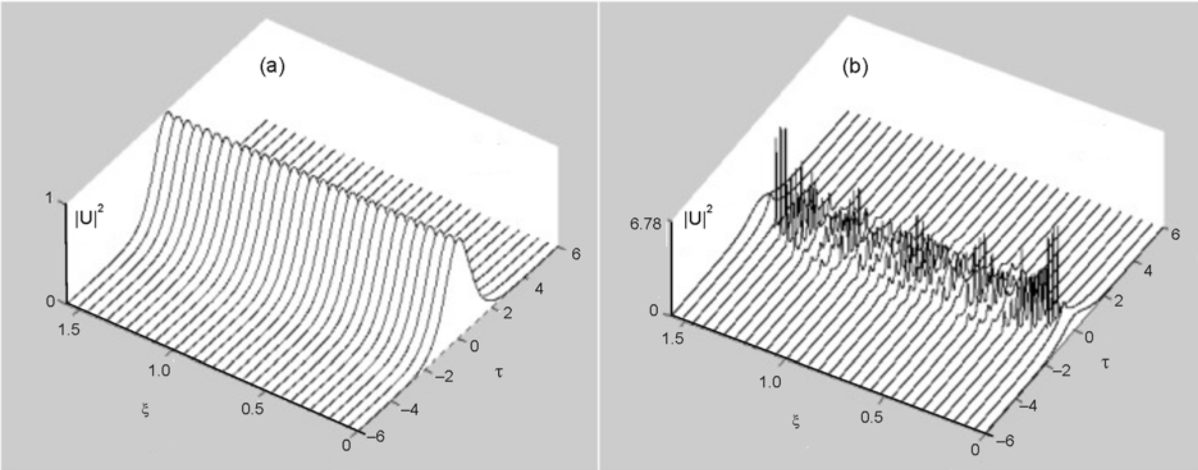

Fig. 7. Change of the pulse intensity in the propagation process for the case of fundamental (a) and tenth-order solitons (b) over one soliton period $\xi = \pi / 2$

envelope function in the location $\xi + \Delta\xi$:

$$U(\xi + \Delta\xi) =$$

$$= F^{-1}\left[V(\xi + \Delta\xi)\exp\left(\left(-\frac{i\omega^2}{2} + i\omega^3\delta_3\right)(\xi + \Delta\xi)\right)\right]. \quad (94)$$

Errors in applying (94) are of orders $(\Delta\xi)^5$. In comparison to calculations performed by (89), formula (94) has a higher accuracy, although the computational time is longer because the number of calculation steps is very large.

In the simulations performed below we have used both of the algorithms presented above and compared the obtained results. They are almost the same when the interval $\Delta\xi$ is relatively small.

At first we compare the numerical simulations performed by using algorithms introduced above with analytical results obtained in some special cases. In this way we test the accuracy of these numerical algorithms. We will compare our results with the results of the NLS equation for the case of picosecond pulses. We consider a very important phenomenon: propagation of the solitons [2, 11].

According to the Inverse Scattering Transform Method, when the higher-order parameters δ_3, S and τ_R in Eq. (25) equal zero and the initial shape of the pulses is the function of secant hyperbolic form, the equation will have the soliton solutions [27, 30]. These solitons exhibit the periodic feature with a characteristic period during propagation. With the exception of the case of the first-order (temporal) soliton (called the fundamental soliton) when the amplitude of the envelope function remains unchanged during propagation, higher-order solitons change in shape and spectrum in a complicated manner, but their shape follows a periodic pattern so that the input shape is recovered at the propagation period $\xi = \pi / 2$. The order of soliton is determined by parameter N in (25). When the value of N is larger (higher-order solitons), the envelope changes in a more complicated way over one soliton period.

We simulated the pulse evolution for the first-order and tenth-order (N = 10) solitons over one soliton period with the input pulse having an initial amplitude [2]:

$$U(0,\tau) = N\,\mathrm{sech}(\tau)\; U\;(0,\,t). \quad (95)$$

Figure 7 shows these results by plotting the pulse intensity $|U(\xi,\tau)|^2$.

In Figure 7(a) the envelope function of the pulse has an unchanged shape in the propagation process conserving the initial form (95). In Figure 7(b) the envelope function has a complex evolution in propagation, but in the end of the period it comes back to the initial shape and this process repeats in the next periods. These results are in good agreement with analytical predictions about the periodic feature in the evolution of the envelope function. Analytical expressions for the higher-order solitons are very complicated and only in the case of the second- and third-order they are explicitly given in literature [11, 27, 30], but for the tenth-order soliton considered above it is presented only by numerical results.

Moving on, we consider the case of multiple soliton propagation. The input amplitude for a soliton pair entering the medium is expressed by

$$U(0,\tau) = \mathrm{sech}(\tau - \tau_1) + r\,\mathrm{sech}\left[r(\tau + \tau_2)\right]\exp(i\theta), \quad (96)$$

where r is the relative amplitude of the two solitons and θ is the relative phase between them [2, 11, 21, 22]. Analytical results [27, 30] show that neighboring solitons either come closer or move apart because of the nonlinear inter-

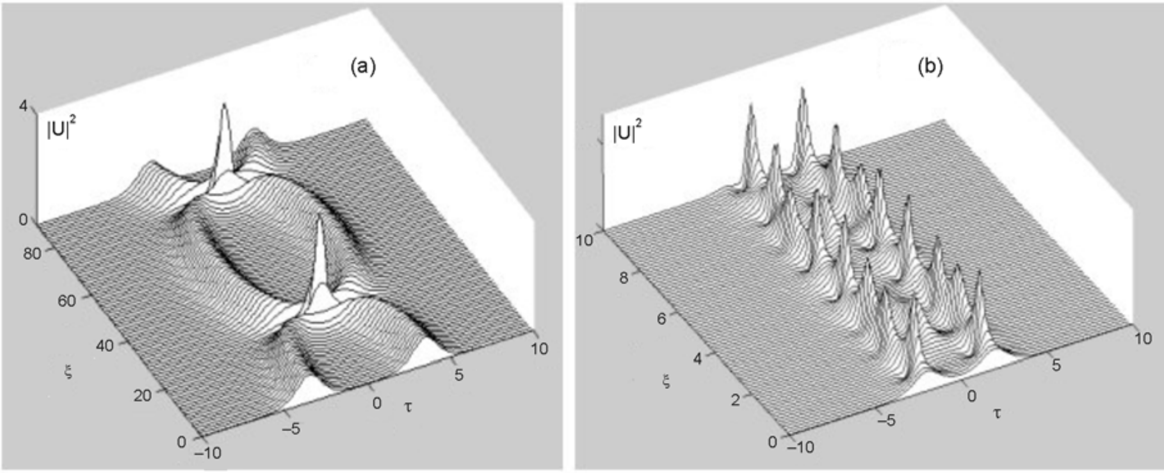

Fig. 8. Collision between two fundamental solitons over the propagation distance $\xi = 90$ (a) and between two second-order solitons over the propagation distance $\xi = 10$ (b)

action between them. The time of soliton collisions depends on both the relative phase θ and the amplitude ratio *r*. Solitons collide periodically along the distance of propagation, the collision period is usually much greater than the soliton period. After the collision the shape of the wave amplitudes remains unchanged and stable. This effect is similar to the collision of the rigid particles, so the name "soliton" reflects the particle feature of the nonlinear waves [2, 11].

The following calculations are performed for the collision between the fundamental solitons and the higher-order solitons. The parameters in (96) are chosen as $r = 1$, $\theta = 0$ (equal-amplitude and in-phase case) and $\tau_1 = \tau_2$ (initial spacing). Numerical results are displayed in Fig. 8.

Figure 8(a) displays the collision process between two fundamental solitons, where $\tau_1 = \tau_2 = 3.5$ and the propagation distance $\xi = 90$. Our results are in good agreement with the calculations in [11, 22].

III.3. Imaginary-time Method

Imaginary time method (ITM) is a powerful tool used to generate stationary states of quantum systems. Here we first describe background of this algorithm for the linear cases and then extend it to the nonlinear situations. Strictly speaking, we have no rigorous proof for the extension but in fact the ITM works very efficiently. In this subsection, we will use notations related to the quantum theory of Bose-Einstein condensates (BECs), but all results could be transferred from atom optics to nonlinear optics by analogy between the propagation equation in the Kerr medium derived above and the Gross-Pitajevski equation for BECs.

For a given Hamiltonian $\hat{H}$ (assumed to be bound from below), the eigenvalue problem is written as follows

$$\hat{H}\varphi_j(x,y) = E_j\varphi_j(x,y). \tag{97}$$

Applying the ITM for that Hamiltonian, we start by introducing an initial wave-function $\Psi_0(x, y)$. The algorithm drives this wave function into the ground state $\varphi_0(x,y)$. The formal expansion of $\Psi_0(x, y)$ in the complete set of $\{\phi_j(x; y)\}$ is

$$\Psi_0(x,y) = \sum_{j=0}^{\infty} a_j\varphi_j(x,y). \tag{98}$$

Assuming $\Psi_0(x,y)$ as wave-function at time $t = 0$, the time evolution is performed by an unitary operator $\hat{U}$, which is acting as

$$\begin{aligned}\Psi(x,y,t) &= \hat{U}(t)\Psi_0(x,y) = \\ &= e^{-\frac{i}{\hbar}\hat{H}t}\sum_{j=0}^{\infty} a_j\varphi_j(x,y) = \\ &= \sum_{j=0}^{\infty} a_j e^{-\frac{i}{\hbar}E_j t}\varphi_j(x,y).\end{aligned} \tag{99}$$

The name of the method originates from the implementing "time evolution" in the imaginary regime $it \Rightarrow \tau$ (*t* is real time)

$$\Psi(x,y,t) = \sum_{j=0}^{\infty} a_j e^{-\frac{E_j\tau}{\hbar}}\varphi_j(x,y). \tag{100}$$

Unlike real time evolution, the "imaginary time evolution" is realized by exponential damping-factors: $E_j\tau/\hbar$. The terms correspond to high energies are damped faster than low energy ones and the ground state is damped least. For

$\tau \to \infty$ all components approach zero; therefore, to avoid this result of a naive, one has to re-normalize the wave function after each time step $\Delta\tau$ (to guarantee the unitary evolution in every step of calculations). By doing so, the wave-function after n time steps reads

$$\Psi(x,y,n\Delta\tau)=\sum_{j=0}^{\infty}\frac{a_j e^{-E_j n\Delta\tau/\hbar}}{\sqrt{\sum_{k=0}^{\infty}|a_k|^2 e^{-2E_k n\Delta\tau/\hbar}}}\varphi_j(x,y). \quad (101)$$

As $E_0=\min\{E_k\}$, the denominator of the above expression behaves like $\sqrt{|a_0|^2 e^{-2E_0 n\Delta\tau/\hbar}}=|a_0|e^{-E_0 n\Delta\tau/\hbar}$ at the limit $n\to\infty$. Hence

$$\lim_{n\to\infty}\Psi(x,y,n\Delta\tau)=\frac{a_0}{|a_0|}\varphi_0(x,y). \quad (102)$$

Thus, this algorithm converges any initial wave function $\Psi_0(x,y)$ to the ground state $\varphi_0(x,y)$.

Now we turn to nonlinear cases. We assume that evolution equation of quantum system takes the form

$$i\Psi_t=-\frac{1}{2}\left(\Psi_{xx}+\Psi_{yy}\right)+U(x,y)\Psi+g|\Psi|^2\Psi, \quad (103)$$

which can be rewritten as

$$\Psi_t=-i\left(\hat{D}+\hat{N}\left[|\Psi|^2\right]\right).\Psi \quad (104)$$

which has the same form as the equation (88). Here $\hat{D}$ and $\hat{N}$ are linear and nonlinear operators respectively given by

$$\hat{D}=-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right), \quad (105)$$

$$\hat{N}\left[|\Psi|^2\right]=U(x,y)+g|\Psi|^2. \quad (106)$$

Applying the ITM for the Eq. (104), we introduce an trial wave-function $\Psi_0(x, y)$ with given norm

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi_0(x,y)|^2\,dxdy=N. \quad (107)$$

Evolution of the system in small interval time Δt is approximated to

$$\Psi(x,y,\Delta t)\approx e^{-\Delta t(\hat{D}+\hat{N})}\Psi_0(x,y)\approx \approx e^{-i\Delta t\hat{D}}e^{-i\Delta t\hat{N}}\Psi_0(x,y). \quad (108)$$

The basic idea of this approximation is that over sufficiently small interval Δt the linear and nonlinear terms can be assumed to act independently.

Similar to the linear case, if we implement "time evolution" in the imaginary regime: $i\Delta t\Rightarrow\Delta\tau$ then the result takes

$$\Psi(x,y,\Delta\tau)\approx e^{-\Delta\tau\hat{D}}e^{-\Delta\tau\hat{N}}\Psi_0(x,y). \quad (109)$$

Again, we observe exponential damping of amplitude of the wave-function. To avoid this fact, we renormalize the wave-function as the following way

$$\tilde{\Psi}(x,y,\Delta\tau)= =\sqrt{\frac{N}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi_0(x,y,\Delta\tau)|^2\,dxdy}}\Psi(x,y,\Delta\tau). \quad (110)$$

This new wave-function is used as $\Psi_0(x, y)$ for evolution in next interval time $\Delta\tau$. In applications, we repeated the calculations (109) and (110) until convergence is reached. Notice that in nonlinear cases the superposition (98) is invalid, therefore we can not state that the obtained wave-function $\Phi(x, y)$ is ground state of the system. In general, one can proof that the Hamiltonian describing the equation (103) is not bound from below. That means the wave-function $\Phi(x; y)$ is a stationary state of the considered system. It corresponds to a fixed point of the Hamiltonian in functional space.

In the next section we will use this numerical method to consider a very interesting phenomenon, namely the collapse of the pulse in the Kerr medium. The formalism of ITM will be used as a test for the variational approach described in II.6.

IV. DYNAMICS OF COLLAPSE OF OPTICAL PULSES IN KERR MEDIUM

Consideration of the self-focusing effect in nonlinear propagation of light is interesting both theoretically and practically. Theoretically, because sometimes we can see dramatic concurrence between different nonlinear effects of the pulse propagation in the nonlinear medium. When the self-focusing effect dominates over the other effects as the dispersion, diffraction etc. the amplitude of the optical pulse (soliton) increases drastically. On the other hand, if we reach a critical point the pulse completely collapses. In practice, this phenomenon is very dangerous from the practical point of view, because it usually destroys the optical material.

As it has been emphasized above, we use here an analogy between nonlinear optics in Kerr media and the Bose-Einstein condensate (BEC) system [4]. As it has been mentioned above, a common ground here is the nonlinear Schrödinger equation, which with the proper substitution of variables describes both types of phenomena. In nonlinear optics it is a light propagation equation that relates the signal at the end of the nonlinear crystal to the signal at the input face of the medium. In the Bose-Einstein condensate dynamics it is the called Gross-Pitajevskii equation. Hence, all results of the consideration in this section can be transferred into the BEC systems [35]. Thus, for some values of the nonlinear coupling constant we can have the collapse and the explosion of the BEC [36]. Such collapse of the self-focusing waves described by the nonlinear Schrödinger equation (NLSE) in nonlinear optics and plasma turbulence has been reviewed in [37]. In this section we use the VM and we will predict the critical point in which the optical pulse collapses. One should keep in mind that when we apply the VM the choice of the trial functions becomes essential. In practice, we shall concentrate on two cases. They correspond to the Gaussian Ansatz (GA) and the Secant Ansatz (SA), as well, and we have performed variational calculations for such two types of the trial functions. Moreover, to confirm numerically our analytical predictions we use the time imaginary method (Subsection III.3). We see that the secant trial function is more proper.

IV.1. Variational approximation

In Section II, using a method based on a consistent and mathematically rigorous expansion of the linear dispersion relation with included nonlinear optical response of the medium, we derived a general propagation equation for light pulse in an arbitrary dispersive nonlinear medium which is called the Generalized Nonlinear Schrödinger Equation (GNLS). In the case of Kerr media we have a well-known cubic nonlinearity which leads to the well-known Nonlinear Schrödinger Equation (NLS). In this paper we concentrate on the model of two-dimensional (2D) NLS. this model describes the propagation of the pulse in a Kerr medium in presence of a harmonic potential [4]:

$$i\Psi_t = -\frac{1}{2}\left(\Psi_{xx} + \Psi_{yy}\right) + \frac{\omega^2 x^2}{2}\Psi - g|\Psi|^2\Psi \tag{111}$$

where g is a nonlinear coefficient. Our notations are chosen for easy transfer of the results to the case of Bose-Einstein condensates. This is a special case of the transformed NLS equation, so-called TNLS equation considered by Berge [37] (see Equation (152) in this paper). The powerful variational method based on choosing a proper trial function. This trial function should, of course, remain compatible with the main invariants and conservation laws of the original NLS equation. To find soliton solutions of the equation given above, corresponding to the propagation of the pulse on the xy plane we use the following SA in two directions x and y:

$$\begin{gathered}\Psi(x,y,t) = A(t)\operatorname{sech}\left(\frac{x}{W(t)}\right)\operatorname{sech}\left(\frac{y}{V(t)}\right)\times \\ \times\ \exp\left\{i\varphi(t) + \frac{1}{2}\left(b(t)x^2 + \beta(t)y^2\right)\right\},\end{gathered} \tag{112}$$

where the variational parameters $A(t), \varphi(t), b(t), \beta(t)$, $W(t)$ and $V(t)$ stand for the amplitude, total phase, spatial chirp coefficients, transverse widths along x and y directions, respectively. Moreover, we use in VA scheme the Lagrange function of the following form [38]:

$$\begin{gathered}L = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdyi\Big[\left(\Psi_t\Psi^* - \Psi_t^*\Psi\right) - \left|\Psi_x\right|^2 + \\ -\left|\Psi_y\right|^2 - \omega^2 x^2|\Psi|^2 + g|\Psi|^4\Big].\end{gathered} \tag{113}$$

Consequently, we apply our ansatz to the above Lagrange function and integrate the result over spatial variables x and y. Finally, we obtain the following function:

$$\begin{gathered}L = 4A^2WV\left[\frac{2gA^2}{9} - \varphi' - \frac{1}{6}\left(\frac{1}{W^2} + \frac{1}{V^2}\right) + \right. \\ \left. -\frac{\pi^2}{24}\left(V^2\left(\beta' + \beta^2\right) + W^2\left(b' + b^2 + \omega^2\right)\right)\right].\end{gathered} \tag{114}$$

At this point we are in the position to derive Euler-Lagrange equations for our parameters (treated as a dynamical variables) in the form of the following system of differential equations:

$$b = \frac{W'}{W}, \tag{115}$$

$$\beta = \frac{V'}{V}, \tag{116}$$

$$W'' = \frac{4}{\pi^2 W^3} - \frac{2g}{3\pi^2 VW^2} - \omega^2 W, \tag{117}$$

$$V'' = \frac{4}{\pi^2 V^3} - \frac{2g}{3\pi^2 WV^2}, \tag{118}$$

$$\varphi' = \frac{g}{9WV} - \frac{1}{6}\left[\frac{1}{V^2} + \frac{1}{W^2} + \frac{\pi^2}{4}\left(\omega^2 W^2 + WW'' + VV''\right)\right], \tag{119}$$

and

$$A^2WV = \text{const.} \tag{120}$$

Moreover, since for our *ansatz* we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi|^2\,dxdy = 4A^2WV \tag{121}$$

we can derive that

$$4A^2WV = 1. \tag{122}$$

At this point we shall concentrate on the stationary regime. Therefore, for the above differential equations we set $W' = W'' = 0$ and $V' = V'' = 0$. As a result, we obtain the following solutions for our parameters:

$$\beta = b = 0, \tag{123}$$

$$V = \frac{6}{g}W, \tag{124}$$

$$W = \sqrt[4]{\frac{36-g^2}{9\pi^2\omega^2}}. \tag{125}$$

One can see that the form of the last formula leads to the necessary condition for the existence of solitons: the value of nonlinearity parameter g should satisfy the inequality $g < 6$. Therefore, we treat the value $g_c = 6$ as a critical one, and when the value of g becomes greater than g_c, the collapse of soliton occurs. For such a case the pulse will becomes very narrow and its amplitude tends to infinity.

To confirm these analytical results we apply by using direct numerical time imaginary method introduced in III.3.

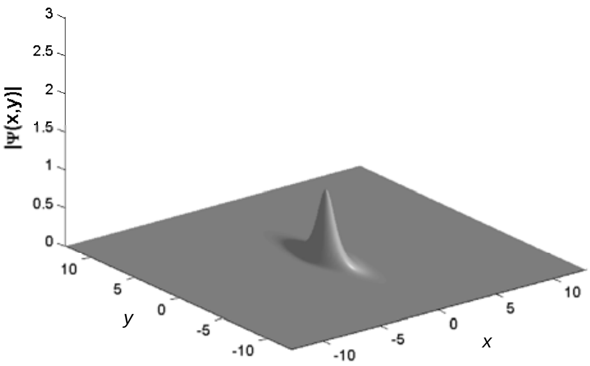

Fig. 9. Soliton solution corresponding to the value of nonlinearity parameter which is below the critical value ($g = 3$). The parameter $\omega = 4$

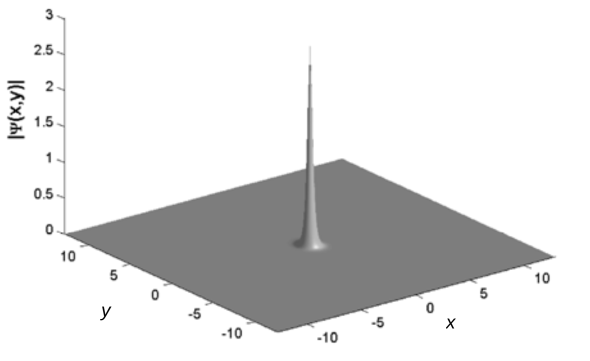

Fig. 10. The same as in Fig. 9 but for g close to its critical value ($g = 5.9$)

The numerical results are shown in Fig. 9, where we have assumed that $g = 3$, $\omega = 4$ and in Fig. 10 for $g = 5.9$ and $\omega = 4$. We see that when the value nonlinear coefficient becomes closed to its critical value (Fig. 10), the pulse becomes very sharp and narrow, contrary to the situation depicted in Fig. 9 (the pulse amplitude tends to infinity when the nonlinear coefficient approaches the critical value).

IV.2. Optical chemical potential

At this point we come back to the soliton solution of our problem. This solution can be expressed as:

$$\Psi(x, y, t) = e^{-i\mu_{2D}t}\Phi(x, y), \tag{126}$$

μ_{2D} is an "optical chemical potential" (OCP) which is strictly related to the energy of the pulse. moreover, we assume that $\Phi(x, y)$ is a real function of spatial variables x and y.

In particular, for the functions discussed here, we substitute expressions (124), (125) determining the widths of the soliton to the Eq. (119) for φ' and we obtain:

$$\varphi' = -\frac{\pi\omega}{18}\frac{18-g^2}{\sqrt{36-g^2}}. \tag{127}$$

Hence the phase of soliton can be expressed as:

$$\varphi(t) = \int \varphi' dt = -\frac{\pi\omega t}{18}\frac{18-g^2}{\sqrt{36-g^2}}. \tag{128}$$

From the above definition of the optical chemical potential (126), after comparing it with our ansatz, we can easily conclude that

$$\varphi(t) = -\mu_{2D}t \tag{129}$$

and therefore, we can write that

$$\mu_{2D} = \frac{\pi\omega}{18}\frac{18-g^2}{\sqrt{36-g^2}}. \tag{130}$$

The dependence of OCP on g and ω is displayed in Fig. 11. We see that its value decreases rapidly as g goes to 6, and moreover, as ω becomes greater and greater such decrease becomes more pronounced for values of g smaller than 6.

In the next step we derive expression for the OCP corresponding to GA. For that purpose we apply the results presented in [38]. Thus, the formula for μ_{2D} can be written in the following form:

$$\mu_{2D} = \frac{\omega}{2\pi}\frac{2\pi^2-g^2}{\sqrt{4\pi^2-g^2}}. \tag{131}$$

It is easily seen that for the case discussed here the critical value of nonlinearity $g_c = 2\pi$.

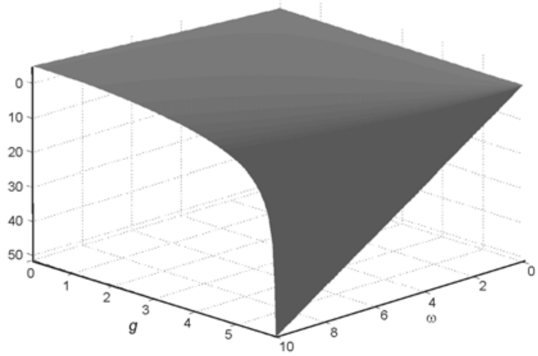

Fig. 11. Chemical potential obtained from VA as a function of g and ω

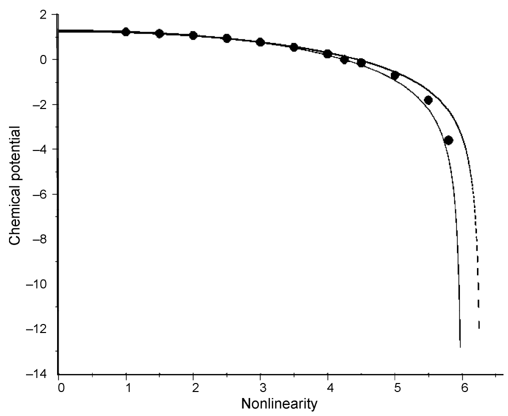

Fig. 12. Chemical potential as a function of the nonlinearity parameter g for $\omega = 2.5$

Now, we shall again confirm our analytical results using numerical calculations. Thus, Fig. 12 shows the values of OCP as a function of g with assumption that the value of ω is fixed $\omega = 2.5$ calculated numerically (circle marks) and derived from our analytical formulas (lines). The dashed line corresponds to GA, whereas the continuous line to the SA. We see that the continuous line is better fitted to the numerical results than its dashed counterpart, so we can conclude that the trial secant function gives more accurate results than the Gaussian one.

IV.3. Dynamics of the collapse

In this section we shall concentrate on the collapse phenomenon. In particular, we will discuss the case when $g \geq 6$. In this regime we have:

$$V = \frac{6}{g} W \approx W. \tag{132}$$

At this point we need to assume that our system has cylindrical symmetry. If we drop the harmonic term, we can write down that

$$W'' = \frac{4}{\pi^2 W^3} - \frac{2g}{3\pi^2 W^3} = \frac{(12-2g)}{3\pi^2 W^3}. \tag{133}$$

Next, if we set that

$$\lambda = \frac{2(g-6)}{3\pi^2} > 0 \tag{134}$$

we get the following ordinary differential equation determining the width $W(t)$:

$$W'' = -\frac{\lambda}{W^3}. \tag{135}$$

We can easily solve this equation assuming that $W(0) = W0$, $W'(0) = 0$ (for stationary solution), where the parameter $W0$ is a certain value of the initial width of the pulse.

Thus, after integrating the equation (135) we obtain the solution

$$W(t) = W_0 \sqrt{1 - \frac{\lambda t^2}{W_0^4}} \tag{136}$$

and we introduce the time of collapse t_{col} by demanding:

$$W(t_{col}) = W \sqrt{1 - \frac{\lambda t_{col}^2}{W_0^4}} = 0. \tag{137}$$

Consequently, we can write that

$$t_{col} = \frac{W_0^2}{\sqrt{\lambda}}. \tag{138}$$

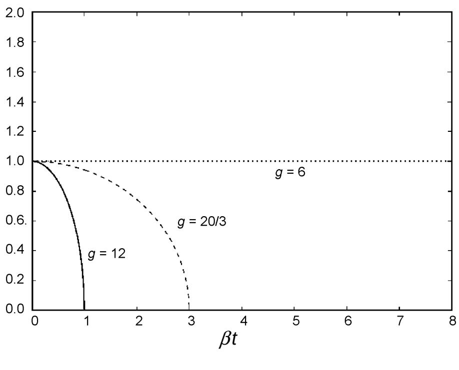

Fig. 13. The time-dependence of the pulse width $W(t)$ for various values of g for the case of collapses. The time t is scaled in the units of β ($\beta = 2/\pi W_0^2$)

Figure 13 shows the change of the pulse width for the case when the collapse occurs. We see that this width decreases to zero for shorter times as the value of the parameter g increases. Moreover, we see that the situation depicted in Fig. 13 agrees with that shown in Fig. 2 on the page 304 of [37].

V. CONCLUSIONS

In this paper we have derived the generalized nonlinear Schrödinger (GNLS) equation for the propagation process of the ultrashort pulses in the Kerr medium. The influence of the higher-order dispersive and nonlinear effects, especially the nonlinear effect induced by the stimulated Raman scattering, have been considered in detail.

Because the GNLS equation is strongly nonlinear, the problem of solving it is a difficult task. We find an exact analytical solution for this equation in the general case by using the developed Jacobi elliptic function expansion. Several approximate methods of solving it are applied. We presented the powerful varational method and three useful numerical methods. Our results calculated by these methods are in good agreement with those obtained before by several authors.

In Sec. IV we have considered TNLS equation concentrating on the special case describing the wave collapses effect. In particular, this equation has been applied to describe light pulse propagation in a Kerr medium. Employing the variational scheme, we have shown that for the two cases discussed (for the secant ansatz and for gaussian one, as well) the wave collapse can appear. Moreover, it was proved that the secant trial function is more proper than its Gaussian counterpart. This fact is quite understandable because the "non-perturbed" solution is the secant hyperbolic solution. It should be stressed out that by the analogy between the propagation equation and Gross-Pitajevski equation for Bose-Einstein condensates we can transfer obtained results to the case of BECs placed in a external harmonic potential.

Acknowledgements

We would like to express our deep gratitude to Marek Trippenbach for pleasant and fruitful collaborations.

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