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Two-Temperature Generalized Thermopiezoelasticity for One Dimensional Problems – State Space Approach

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Abstract: The theory of two-temperature generalized thermoelasticity, based on the theory of Youssef is used to solve boundary value problems of one dimensional piezoelectric half-space with heating its boundary with different types of heating. The governing equations are solved in the Laplace transform domain by using state-space approach of the modern control theory. The general solution obtained is applied to a specific problems of a half-space subjected to three types of heating; the thermal shock type, the ramp type and the harmonic type. The inverse Laplace transforms are computed numerically using a method based on Fourier expansion techniques. The conductive temperature, the dynamical temperature, the stress and the strain distributions are shown graphically with some comparisons.

Key words: piezoelectric material, generalized thermoelasticity, state space, two-temperature

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Nomenclature:

A_{ij} – the components of relaxation time
a – the two-temperature parameter
C_E – specific heat at constant strain
c_{ijkl} – the elastic coefficients
c_0 – $\sqrt{(\lambda+2\mu)/\rho}$ – longitudinal wave speed
D_i – the components of electric displacement
d_i – the pyroelectric coefficient
E_i – the components of electric field vector
e_{ij} – the components of strain tensor
h_{ijk} – the piezoelectric coefficients
k_{ij} – the components of thermal conductivity
q_i – the components of the heat flux vector
T – absolute temperature
T_0 – reference temperature
t – time
t_0 – ramping time parameter
u_i – components of displacement vector
v_i – the electric potential function
α – $\gamma T_0/(\lambda+2\mu)$ – dimensionless thermoelastic coupling constant
α_T – coefficient of linear thermal expansion
β_{ik} – the components of dielectric tensor
γ – $(3\lambda+2\mu)\alpha_T$
Ω – the angular frequency of thermal vibration
δ_{ij} – Kronecker delta function
ε – $\gamma/\rho C_E$ – dimensionless mechanical coupling constant
ζ – the entropy
η – $\rho C_E/k$ – the thermal viscosity
θ $(T-T_0)$ – the dynamical temperature increment such that $|T-T_0|/T_0 \ll 1$
λ, μ – Lamé's constants
ρ – density
σ_{ij} – components of stress tensor
σ – the principal Stress component
τ_0 – one relaxation time parameter
φ – the conductive temperature
ω – dimensionless two-temperature parameter

I. INTRODUCTION

For classical uncoupled and coupled theories of thermoelasticity, the heat conduction equations are of the diffusion types which lead to infinite speeds of propagation for heat waves contrary to physical observations. Widespread attention to eliminate this paradox has been given to thermoelasticity theories which admit a finite speed for the propagation of thermal waves. Many authors have formulated generalized theories involve a hyperbolic-type heat equation and are referred to as generalized thermoelasticity. Three generalizations to the coupled theory were introduced. The first theory was developed by Lord and Shulman [1]. In this theory they obtained a wave-type heat equation by modifying Fourier's law to contain the heat flux vector as well as its derivative and include one relaxation time. Since the heat equation of this theory is of the wave-type, it ensures finite speed of propagation for heat and elastic waves [2]. This eliminates the paradox accompanying the infinite speed of heat propagation in the classical theory and allows for the so called second sound effects in solids. Ignaczak contributes to the thermoelasticity with one relaxation by the proofs of uniqueness theorems under different conditions [3].

The second generalization to the coupled theory of elasticity is what is known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate dependent thermoelasticity. In this theory an entropy production inequality was proposed by Müller [4]. It represents restrictions on a class of constitutive equations. The Green and Lindsay (G-L) theory modifies both the energy equation and the Duhamel-Neuman relation and admits two relaxation times and modify all equations of the coupled theory, not only the heat equation. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry [5, 6]. They also obtained another version of this constitutive equations.

In the third theory [C-T] theory, Tzou [7] replace the Fourier law by an approximation to a modification of the Fourier law with two different translations for the heat flux and the temperature gradient. This theory is known as the dual-phase-lag thermoelasticity.

Piezoelectricity is the phenomenon whereby electric polarization is devoted in deformed materials. The effect of electro-mechanical coupling in such materials has immense potential in engineering applications. A good example is the use of this class of materials as sensors and actuators in micro- electro-mechanical system (MEMS), for instance, the piezoelectric accelerometer which triggers an airbag in ten of second during accident.

With the advent of new generation of electronic devices, their reliability and integrity are essential for safe operation. In addition, because these devices are to operate under various electro-thermo-mechanical conditions over a broad spectrum, their design and manufacturing represent a great challenge in engineering. In view of its versatility and important to engineering applications, we devoted our attention to piezoelectric ceramic which have been extensively used in many engineering applications. So, the theory of generalized thermo-piezoelectricity has been the object of numerous investigations in the last decades or so, concerning both its theoretical foundations and the applications.

The theory of generalized thermoelasticity was extended so as to involve electromagnetic media, piezoelectric in particular. The theory of thermo-piezoelectricity was first proposed by Mindlin [8]. He also derived governing equations of a thermo-piezoelectric plate [9]. Nowacki [10, 11] has studied the physical laws for the thermo-piezoelectric materials. Chandrasekharaiah [12] has generalized Mindlin's theory of thermo-piezoelectricity to account for the finite speed of propagation of thermal disturbances. Majhi [13] studied the transient thermal response of a semi-infinite piezoelectric rod subjected to a local heat source along the length direction, by introducing a potential function and applying the L-S theory. Sharma and Kumar [14] studied plane harmonic waves in piezo-thermoelastic materials. Bassiouny and Ghaleb have solved a one dimensional problem in the generalized theory of thermopiezoelasticity [15]. Tianhu et al. [16, 17] discussed various thermal shock problems of piezoelectric plate by applying the L-S and G-L theories. Baljeet [18] used Green-Lindsay and Lord-Shulman theories for generalized thermo-piezoelectric solid to study the plane waves in a two-dimensional model.

Recently, Youssef [19] has improved the previous theories of the generalized thermoelasticity (G-L) and (L-S) with a new theory which depends on two distinct temperatures, that is, the conductive temperature and the thermodynamic temperature, this theory is called; Theory of two-temperature generalized thermoelasticity (Y-TTGTE). The uniqueness solution of the last theory has been derived also by Youssef.

The present work, use the state space approach and the two-temperature theory (Y-TTGTE) to study the effect of the presence of the heat conduction φ in Fourier's law instead of the usual thermodynamic temperature on the behavior of the solutions in generalized thermo-piezoelectricity. Considering the (Y-TTGTE) model, the governing differential equations for two-temperature generalized thermo-piezoelectric solid are formulated. Laplace transform technique is applied to thermal shock of semi infinite piezoelectric rod to obtain the solution in the transformed domain, in combination with a numerical inversion formula.

II. FORMULATION OF THE PROBLEM

Consider a semi-infinite piezoelectric rod occupying the interval $x \geq 0$. At the near end of the rod, a thermal effect is given which raises the temperature of this end to a prescribed temperature with known function and free of stress. Piezoelectric rod direction be parallel with the axial direction. The corresponding boundary conditions may be written as follows [20]:

$$\theta(x,t)\Big|_{x=0} = F(t), \quad \sigma(x,t)\Big|_{x=0} = 0 \tag{1}$$

In the absence of body force, free charge and inner heat sources, the generalized thermo-piezoelectric governing differential equations as follow:

Equations of motion:

$$\sigma_{ij,j} = \rho\, \ddot{u}_i \,, \tag{2}$$

Equation of entropy production (in the absence of inner heat source):

$$q_{i,i} = -T_0\, \dot{\zeta} \,, \tag{3}$$

Stress-strain-temperature:

$$\sigma_{ij} = c_{ijkl}\, e_{kl} - h_{ijk}\, D_k - \beta_{ij}\, \theta, \tag{4}$$

Gauss equation and electric field relation:

$$D_{i,i} = 0 \,, \tag{5}$$

$$E_i = -\, v_{,i} \,, \tag{6}$$

$$E_i = h_{ikl}\, e_{kl} + \tau_{ik}\, D_k - d_i\, T \,. \tag{7}$$

Equation of entropy:

$$\zeta = \beta_{ij}\, e_{ij} + d_i\, D_i + c\, T \,. \tag{8}$$

Strain-displacement relations:

$$e_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \,. \tag{9}$$

We will assume the following new form of the heat conduction equation [19]:

$$q_i + A_{ij}\, \dot{q}_{,j} = -\, K_{ij} \varphi_{,j} \,, \tag{10}$$

where φ is the conductive temperature and satisfies the relation

$$\varphi - T = a\, \varphi_{,ii} \tag{11}$$

in which $a > 0$ is the two-temperature parameter and k_{ij} is the components of thermal conductivity tensor.

Once more, we indicate to the fact that the new modification to the usual theory of generalized thermo piezoelectric is the presence of the conductive temperature into Fourier's law of heat conduction.

In the above equations, a comma followed by a suffix denotes material derivatives and a superposed dot denotes the derivatives with respect to time.

III. ONE DIMENSION FORMULATION

For one-dimensional problem we assume displacement component of the form

$$u_x = (x,t) \,, \quad u_y = u_z = 0 \,. \tag{12}$$

The following are the linearized basic equations in one-dimensional formulation:

$$(\lambda + 2\mu)\, \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial \theta}{\partial x} = \rho\, \frac{\partial^2 u}{\partial t^2} \,, \tag{13}$$

$$\sigma = (\lambda + 2\mu)\, \frac{\partial u}{\partial x} - \gamma\theta - h\, D \,, \tag{14}$$

$$k\, \frac{\partial^2 \varphi}{\partial x^2} = \\ = \left(\frac{\partial}{\partial t} + \tau_0\, \frac{\partial^2}{\partial t^2} \right) \left[\rho\, C_E\, \theta + \gamma\, T_0\, e \right] , \tag{15}$$

$$\varphi - T = a \frac{\partial^2 \varphi}{\partial x^2} \,, \tag{16}$$

$$e = \frac{\partial u}{\partial x} \,, \tag{17}$$

$$\frac{\partial D}{\partial x} = 0 \,, \tag{18}$$

$$E_1 = -\frac{\partial v}{\partial x} \,, \tag{19}$$

where, $\gamma = \alpha_t (3\lambda + 2\mu)$, α_t is the coefficient of the linear thermal expansion and *x* is the coordinate taken along the rod, measured from the finite end.

It is convenient now to introduce the following dimensionless variables:

$$u' = c_0\, \eta\, u \,, \quad t' = = c_0^{\,2}\, \eta\, t \,, \quad \sigma' = \frac{\sigma}{(\lambda + 2\mu)} \,,$$

$$\theta' = \frac{T - T_0}{T_0}, \quad \tau' = c_0^{\,2}\,\eta\,\tau\,, \quad \varphi' = \frac{\varphi - T_0}{T_0}$$

$$D' = \frac{h}{\lambda + 2\mu}\,D\,, \quad \eta = \frac{\rho\,C_E}{k}\,, \quad c_0^2 = \frac{\lambda + 2\,\mu}{\rho}\,,$$

$$t_0' = = c_0^{\,2}\,\eta\;t_0\,, \quad \Omega' = = \frac{\Omega}{c_0^{\,2}\,\eta}\,. \tag{20}$$

From Gauss's law, since there is no free charge inside the rod, we have $\partial D/\partial x = 0$ then it follows that:

$$D(t) = \text{const.} \tag{21}$$

Substituting Eq. (16) into Eqs. (12)-(19) and dropping the primes for convenience, we obtain the following set of non-dimensional equations:

$$\frac{\partial^2 e}{\partial x^2} - \alpha\,\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 e}{\partial t^2}\,, \tag{22}$$

$$\sigma = e - \alpha\theta - D\,, \tag{23}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau_0\,\frac{\partial^2}{\partial t^2}\right)\left(\theta + \varepsilon e\right), \tag{24}$$

and the following relation between the conductive temperature and the thermo dynamical one:

$$\theta = \varphi - \omega\frac{\partial^2 \varphi}{\partial x^2} \tag{25}$$

where,

$$\alpha = \frac{\gamma T_0}{(\lambda + 2\mu)}\,, \qquad \varepsilon = \frac{\gamma}{\rho\,C_E}\,, \qquad \omega = a\,c_0^2\,\eta^2\,.$$

We assume that, the half space $x \geq 0$ is set to be initially at rest and has reference temperature T_0 such that the initial conditions are assume to be:

$$e(x,0) = \dot{e}(x,0) = 0\,, \tag{26}$$

$$\theta(x,0) = \dot{\theta}(x,0) = 0\,, \tag{27}$$

$$\varphi(x,0) = \dot{\varphi}(x,0) = 0\,. \tag{28}$$

We consider the half-space $x \geq 0$ at a uniform temperature T_0 with its boundary $x = 0$ subjected to heating with general function $F(t)$ and traction free, so that the boundary conditions take the following forms:

$$\varphi_0(0,t) = F(t)\,, \tag{29}$$

$$\sigma(0,t) = 0\,, \tag{30}$$

and that

$$e(x,t) \to 0\,,$$

$$\theta(x,t) \to 0 \text{ and } \varphi(x,t) \to 0 \text{ as } x \to \infty, \quad t > 0\,,$$

Applying Laplace transform defined by:

$$L\{f(t)\} = \overline{f(s)} = \int_0^\infty e^{-st} f(t)\,dt\,, \tag{31}$$

to the both sides of Eqs. (22)-(25), we obtain:

$$\frac{d^2\,\bar{e}}{d\,x^2} - \alpha\,\frac{d^2\,\bar{\theta}}{d\,x^2} = s^2\bar{e}\,, \tag{32}$$

$$\bar{\sigma} = \bar{e} - \alpha\bar{\theta} - \frac{D}{s}\,, \tag{33}$$

$$\frac{d^2\,\bar{\varphi}}{d\,x^2} = (s + \tau_0\,s^2)\bar{\theta} + \varepsilon(s + \tau_0\,s^2)\,\bar{e}\,, \tag{34}$$

$$\bar{\theta} = \bar{\varphi} - \omega\frac{d^2\bar{\varphi}}{d\,x^2}\,, \tag{35}$$

$$\bar{\varphi}(0,s) = \bar{F}(s)\,, \tag{36}$$

$$\bar{\sigma}(0,s) = 0\,, \tag{37}$$

where s denotes the complex argument related to the Laplace transform.

Eliminating $\bar{\theta}$ between Eqs. (34) and (35), we left with:

$$\frac{d^2\,\bar{\varphi}}{d\,x^2} = L\,\bar{\varphi} + L\varepsilon\,\bar{e}\,, \tag{38}$$

where

$$L = L(s) = \frac{s + \tau_0 s^2}{1 + \omega\left(s + \tau_0 s^2\right)}\,.$$

Substituting from Eq. (38) into Eq. (35), we obtain

$$\bar{\theta} = \left(1 - \omega L\right)\bar{\varphi} - \omega L\varepsilon\,\bar{e}\,. \tag{39}$$

Eliminating $\bar{\theta}$ between Eq. (32) and Eq. (33) we obtain

$$\frac{d^2\bar{e}}{d\,x^2} = M\,\bar{\varphi} + N\,\bar{e}\,, \tag{40}$$

where

$$M = M(s) = \frac{\alpha\,L(1 - \omega L)}{1 + \omega\alpha\,\varepsilon\,L}$$

and $$N = N(s) = \frac{s^2 + \alpha\varepsilon L(1-\omega L)}{1+\omega\alpha\varepsilon L}.$$

IV. STATE SPACE APPROACH

Equations (38) and (40) can be written matrix differential equations as follows [21]:

$$\frac{d^2\overline{V}(x, s)}{dx^2} = A(s)\overline{V}(x, s), \tag{41}$$

where

$$\overline{V}(x, s) = \begin{bmatrix} \overline{\varphi} \\ \overline{e} \end{bmatrix}$$

is the state vector in the transform domain, and, $A(s)$ is a 2×2 matrix assume the form

$$A(s) = \begin{bmatrix} L & \varepsilon L \\ M & N \end{bmatrix}. \tag{42}$$

The general solution of Eq. (42) can be obtained in the form:

$$\overline{V}(x, s) = \exp\left[-\sqrt{A(s)}\,x\right]\overline{V}(0, s), \tag{43}$$

where for bounded solution with large x, we have canceled the exponential part that has a positive power, the matrix exponential $\exp(-\sqrt{A(s)}\,x)$ is the transfer matrix and

$$\overline{V}(0, s) = \begin{bmatrix} \overline{\varphi}_0 \\ \overline{e}_0 \end{bmatrix}. \tag{44}$$

where

$$\overline{\varphi}_0 = \overline{F}(s), \tag{45}$$

and from Eqs. (33), (37), (39) and (45), we have

$$\overline{e}_0 = \frac{1}{1+\omega L\varepsilon\alpha}\left[\alpha(1-\omega L)\overline{F}(s) + \frac{D}{s}\right],$$

where

$$\overline{e}_0 = \overline{e}(0, s)$$

The characteristic equation corresponding to the matrix A assumes the form:

$$\lambda^2 - (L+N)\,\lambda + (LN - \varepsilon LM) = 0, \tag{46}$$

The roots of this equation, namely, λ_1 and λ_2, satisfy the following relations

$$\lambda_1 + \lambda_2 = L + N, \tag{47}$$

$$\lambda_1\lambda_2 = LN - \varepsilon LM. \tag{48}$$

The Taylor series expansion for the matrix exponential $\exp(-\sqrt{A(s)}\,x)$ is given by

$$\exp[-A(s)x] = \sum_{n=o}^{\infty} \frac{\left[-\sqrt{A(s)}x\right]^n}{n!}. \tag{49}$$

Using Cayley-Hamilton theorem, we can express A^2 and higher orders of the matrix A in terms of I, A, where I is the unit matrix of second order.

Thus, the infinite series in Eq. (49) can be reduced to the following form

$$\exp\left[-\sqrt{A(s)}x\right] = a_0(x,s)I + a_1(x,s)A(s). \tag{50}$$

where a_0 and a_1 are some coefficients depending on s and x.

By Cayley-Hamilton theorem, the characteristic roots λ_1 and λ_2 of the matrix A must satisfy Eq. (50), thus we have

$$\exp\left(-\sqrt{\lambda_1}\,x\right) = a_0 + a_1\lambda_1, \tag{51}$$

and

$$\exp\left(-\sqrt{\lambda_2}\,x\right) = a_0 + a_1\lambda_2. \tag{52}$$

Solving the above linear system of equations, we get

$$a_0 = \frac{\lambda_1 e^{-\sqrt{\lambda_2}x} - \lambda_2 e^{-\sqrt{\lambda_1}x}}{\lambda_1 - \lambda_2}, \tag{53}$$

and $$a_1 = \frac{e^{-\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_2}x}}{\lambda_1 - \lambda_2}. \tag{54}$$

Hence, we have

$$\exp\left[-\sqrt{A(s)}x\right] = L_{ij}(x,s),\ i,j = 1, 2, \tag{55}$$

where

$$L_{11} = \frac{(\lambda_1 - L)e^{-\sqrt{\lambda_2}x} - (\lambda_2 - L)e^{-\sqrt{\lambda_1}x}}{\lambda_1 - \lambda_2},$$

$$L_{12} = \frac{gL\left(e^{-\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_2}x}\right)}{\lambda_1 - \lambda_2},$$

$$L_{22} = \frac{(\lambda_2 - N)e^{-\sqrt{\lambda_1}x} - (\lambda_1 - N)e^{-\sqrt{\lambda_2}x}}{\lambda_2 - \lambda_1},$$

$$L_{21} = \frac{M\left(e^{-\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_2}x}\right)}{\lambda_1 - \lambda_2}.$$

We can write the solution in (43) in the following form

$$\overline{V}(x,s) = L_{ij}\overline{V}(0,s). \tag{56}$$

Hence, we obtain

$$\bar{\varphi} = \varphi_1 \, e^{-\sqrt{\lambda_2}\,x} - \varphi_2 \, e^{-\sqrt{\lambda_1}\,x}, \tag{57}$$

where

$$\varphi_1 = \frac{(\lambda_1 \bar{\varphi}_0 - L\bar{\varphi}_0 - gL\bar{e}_0)}{\lambda_1 - \lambda_2}, \quad \varphi_2 = \frac{(\lambda_2 \bar{\varphi}_0 - L\bar{\varphi}_0 - gL\bar{e}_0)}{\lambda_1 - \lambda_2},$$

and

$$\bar{e} = e_1 \, e^{-\sqrt{\lambda_2}\,x} - e_2 \, e^{-\sqrt{\lambda_1}\,x}, \tag{58}$$

where

$$e_1 = \frac{(\lambda_1 \bar{e}_0 - M\bar{\varphi}_0 - N\bar{e}_0)}{\lambda_1 - \lambda_2}, \quad e_2 = \frac{(\lambda_2 \bar{e}_0 - M\bar{\varphi}_0 - N\bar{e}_0)}{\lambda_1 - \lambda_2}.$$

Using Eqs. (57) and (58) into Eq. (39), we obtain

$$\bar{\theta} = \theta_1 \, e^{-\sqrt{\lambda_2}\,x} - \theta_2 \, e^{-\sqrt{\lambda_1}\,x}, \tag{59}$$

where

$$\theta_1 = (1-\omega L)\varphi_1 - \omega L e_1,$$

$$\theta_2 = (1-\omega L)\varphi_2 - \omega L e_2.$$

Now, we can get the stress equation by using Eqs. (58) and (59) into Eq. (33), thus we have

$$\bar{\sigma} = \sigma_1 \, e^{-\sqrt{\lambda_2}\,x} - \sigma_2 \, e^{-\sqrt{\lambda_1}\,x} - \frac{D}{s}, \tag{60}$$

where

$$\sigma_1 = (e_1 - \alpha\theta_1), \quad \sigma_2 = (e_2 - \alpha\theta_2).$$

Which complete the solution on the Laplace transform domain.

Application I (Thermal shock problem)

We consider the half-space $x \geq 0$ at a uniform temperature T_0 with its boundary $x = 0$ subjected to thermal shock as follows [17]:

$$F(t) = F_0 \, H(t), \tag{61}$$

where F_0 is constant represent the strength of the shock on the boundary, and $H(t)$ is the Heavyside unit step function.

After using the Laplace transform, we have

$$\bar{\varphi}_0 = \bar{F}(s) = \frac{F_0}{s}, \tag{62}$$

and

$$\bar{e}_0 = \frac{1}{1+\omega L \varepsilon \alpha}\left[\frac{\alpha(1-\omega L)F_0}{s} + \frac{D}{s}\right]. \tag{63}$$

Thus, we get the complete solution for this application on the Laplace transform domain by using Eqs. (62) and (63) into Eqs. (57)-(60).

Application II (Ramp-type heating)

We consider the half-space $x \geq 0$ at a uniform temperature T_0 with its boundary $x = 0$ subjected to thermal shock as follows [20]:

$$F(0,t) = \begin{cases} 0 & t \leq 0 \\ \dfrac{F_0}{t_0} t & 0 < t \leq t_0 \\ F_0 & t > t_0 \end{cases}, \tag{64}$$

where F_0 is constant and t_0 is the ramp type parameter.

After using the dimensionless and the Laplace transform defined previously, we have

$$\varphi_0 = \bar{F}(s) = \frac{F_0\left(1-e^{-s t_0}\right)}{t_0 \, s^2}, \tag{65}$$

and

$$\bar{e}_0 = \frac{1}{1+\omega L \varepsilon \alpha}\left[\frac{\alpha F_0 (1-\omega L)\left(1-e^{-s t_0}\right)}{t_0 \, s^2} + \frac{D}{s}\right]. \tag{66}$$

Thus, we get the complete solution for this application on the Laplace transform domain by using Eqs. (65) and (66) into Eqs. (57)-(60).

Application III (Harmonically varying temperature)

We consider the half-space $x \geq 0$ at a uniform temperature T_0 with its boundary $x = 0$ subjected to thermal shock as follows [22]:

$$F(0,t) = F_o e^{i\Omega t}, \tag{67}$$

where F_0 is constant, Ω is the angular frequency of thermal vibration and $i = \sqrt{-1}$.

After using the dimensionless and the Laplace transform defined previously, we have

$$\varphi_0 = \bar{F}(s) = \frac{F_0}{s - i\Omega}, \tag{68}$$

$$\bar{e}_0 = \frac{1}{1+\omega L \varepsilon \alpha}\left[\frac{\alpha F_0 (1-\omega L)}{s - i\Omega} + \frac{D}{s}\right]. \tag{69}$$

Thus, we get the complete solution for this application on the Laplace transform domain by using Eqs. (68) and (69) into Eqs. (57)-(60).

V. NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

In order to invert the Laplace transform, we adopt a numerical inversion method based on a Fourier series expansion [23, 24].

By this method the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t)=\frac{e^{ct}}{t_1}\left[\frac{1}{2}\bar{f}(c)+R1\sum_{k=1}^{N}\bar{f}\left(c+\frac{ik\pi}{t_1}\right)\exp\left(\frac{ik\pi t}{t_1}\right)\right],$$

$$0<t_1<2t,$$

where N is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that

$$\exp(ct)\ R1\left[\bar{f}\left(c+\frac{iN\pi}{t_1}\right)\exp\left(\frac{iN\pi t}{t_1}\right)\right]\leq\varepsilon_1,$$

where ε_1 is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\bar{f}(s)$. The optimal choice of c was obtained according to the criteria described in [24].

VI. NUMERICAL RESULTS AND DISCUSSION

The numerical values of the thermal temperature, the dynamical temperature, stress and strain have been calculated for small time $t=0.25$, for wide range of $x=0.0$ up to $x=2.0$, and for $\tau=0.05$ for the relaxation time. In the calculation process, the following constants are necessary to be known including $F_0=1.0$, $\varepsilon=0.003887$, $\alpha=0.036991$, $\Omega=10^{-5}$, $\omega=0.1$, $D=10^{-7}$ [16]. The numerical results are displayed graphically.

We have three groups of graphs where we have three applications:

The first group (Figs. 1-4) displays the solution of the problem for the thermal shock. It shows the differences between the theory of one temperature generalized thermoelasticity and the theory of two-temperature generalized thermoelasticity. Results of this group are also compared with the correspondence case of Tianhu.

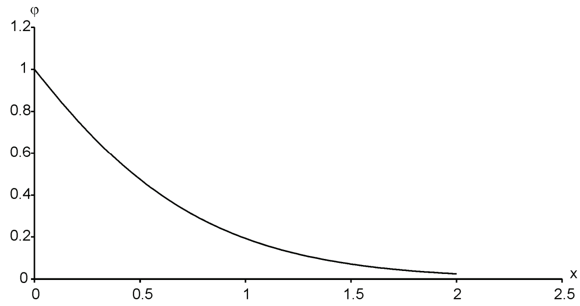

Fig. 1. Heat conduction distribution for thermal shock

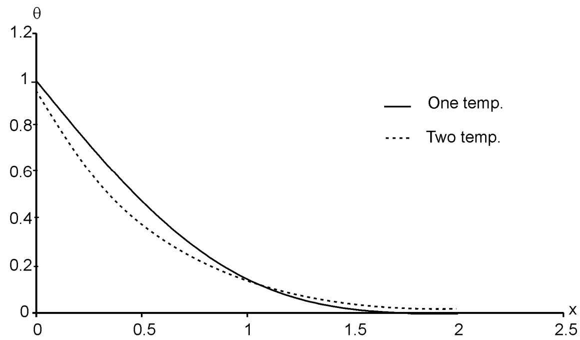

Fig. 2. Dynamical heat conduction distribution for thermal shock

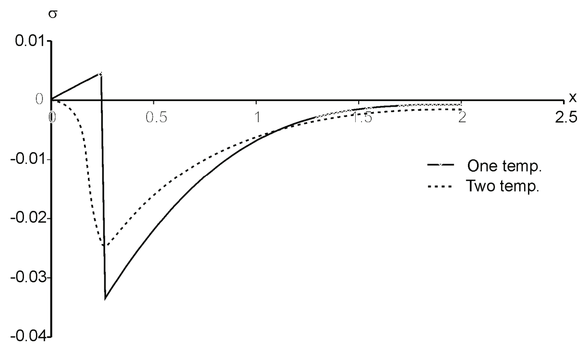

Fig. 3. Stress distribution for thermal shock

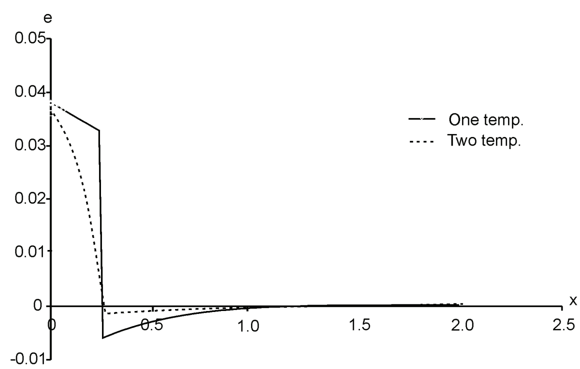

Fig. 4. Strain distribution for thermal shock

1. Figure 1 displays the conductive temperature and we can deduce that, the wave has a finite speed of propagation. This result shows that the two type temperature model agree with the generalized thermoelasticity.
2. From Fig. 2 we deduce that both type of temperature vanish smoothly far from the nearest end of the rod which is more realistic than the correspondence result of Tianhu which shows that the temperature reduced suddenly to zero.
3. Figure 3 displays a comparison of the stress in the context of the two theories. At $x = 0$ the stress reduces to zero which agree with the boundary condition. Comparison with Tianhu work we deduce that the stress distribution is negative in Tianhu work while in the present work we observe that stress in the case of two type temperature is also negative. In the case of one temperature only one jump occur at $x = 0.24$ and the magnitude of the stress is 0.0043276 while in the Tianhu work two jumps occur; one of these occurs at $x = 0.24$. The two temperature parameter remove the discontinuities appear in the correspondence result of Bassiouny and Tianhu.
4. Figure 4 displays a comparison of the strain in the context of the two theories. We have found that, in the theory of Lord and Shulman the strain distribution has a discontinuous as the stress, while in the theory of Youssef, the strain distribution is continuous and smooth.

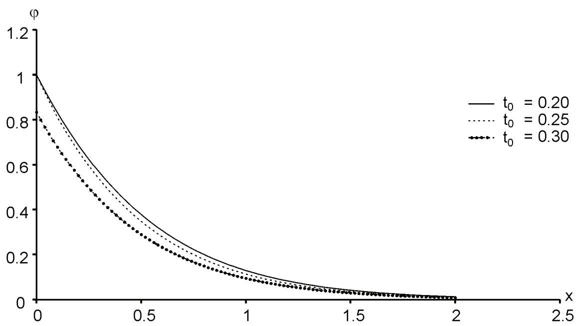

Fig. 5. Heat conduction distribution for ramp-type heating

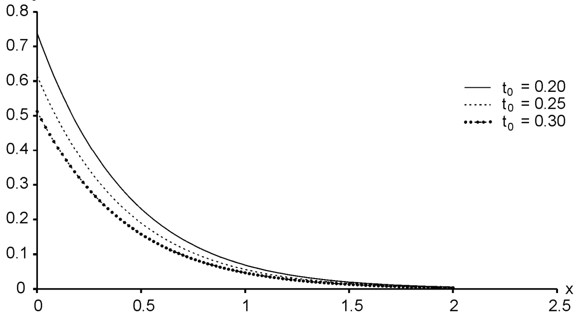

Fig. 6. Dynamical heat distribution for ramp-type heating

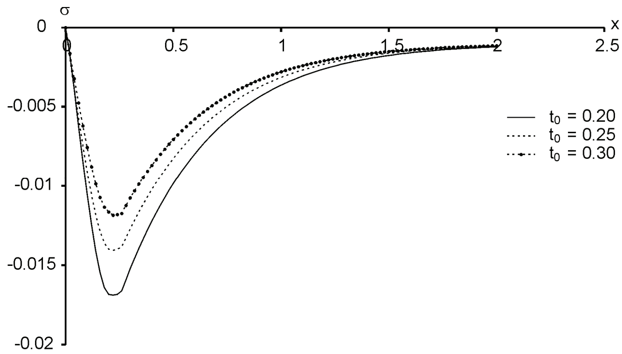

Fig. 7. Stress distribution for ramp-type heating

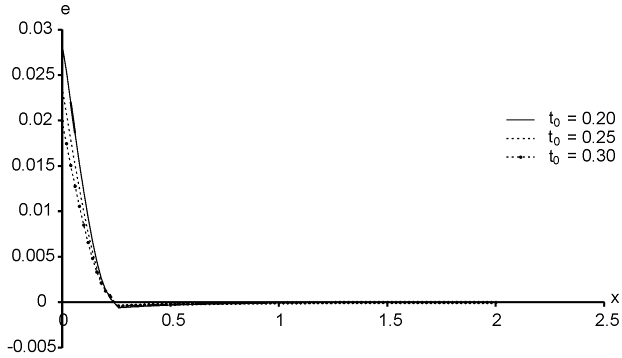

Fig. 8. Strain distribution for ramp-type heating

The second group (Figs. 5-8) displays the solution of the problem for the ramp type heating in the context of Youssef model. This group shows the effect of the ramp type parameter on the results and we found that both type of temperature changes with the same manner due to the changes of the ramp parameter.

The third group (Figs. 9-12) displays the solution of the problem for harmonic heating in the context of Youssef model. This group shows the effect of the relaxation time parameter on the results and we noticed that decreasing the relaxation time increases the heat conductive, the dynami-

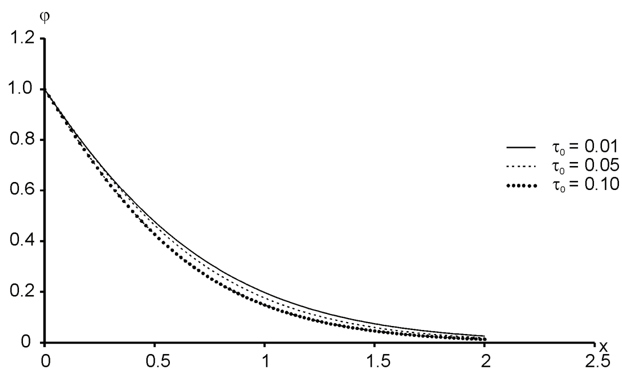

Fig. 9. Heat conduction distribution for harmoically heating

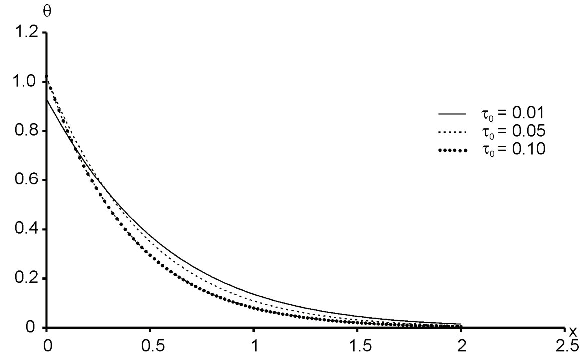

Fig. 10. Dynamical heat conduction distribution for harmoically heating

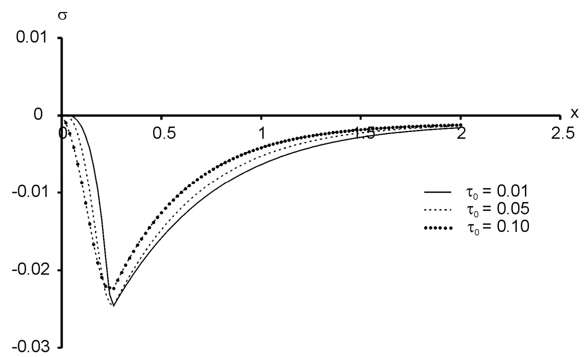

Fig. 11. Stress distribution for harmoically heating

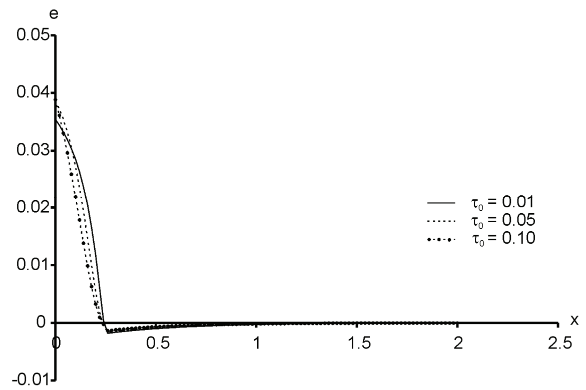

Fig. 12. Strain distribution for harmoically heating

cal heat, the strain and the absolute value of the maximum stress.

VII. CONCLUSION

Due to the application of two type temperature method to the shock problem of piezoelectric material the discontinuities in the stress and strain function have been removed. The way of vanishing the temperature in the present work in comparison with Tianhu work leads us to claim that the present model is more realistic than that of Tianhu. We have found that, the ramp parameter t_0 as well as the relaxation time τ_0 has significant effects on all the fields.

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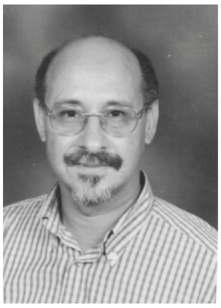

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