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ON REPRESENTATIONS OF COEFFICIENTS IN IMPLICIT INTERVAL METHODS OF RUNGE-KUTTA TYPE

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Abstract: The paper presents one-, two- and three-stage implicit interval methods of Runge-Kutta type for solving the initial value problem. In our previous papers [1] and [2] it was shown that the exact solution belongs to the interval-solution obtained by both kinds of these methods. We continue the problem on the minimization of the widths of interval-solutions.

1. INTRODUCTION

Interval methods for solving the initial value problem in floating-point interval arithmetic give solutions in the form of intervals which contain all possible numerical errors (see [3] or [4]). The estimations of diameters of interval-solutions are possible on account of (see [5]):

- the minimization (with respect to the coefficients) of the interval extension of the principal part of the approximation error (see e.g. [6]),
- the minimization of some constants which occur in the estimation of interval-solutions (see [1] and [2])
- the coefficients of the particular methods which have exact representations in the computer.

In this paper we consider the last of these cases for one-, two- and three-stage implicit interval methods of Runge-Kutta type.

The paper is organized as follows. Section 2 contains the implicit classical methods of Runge-Kutta type for solving the initial value problem. In Section 3 the implicit interval Runge-Kutta methods are given. Section 4 deals with all possible forms of numbers which are exactly represented in floating-point arithmetic. In Section 5 some approximations of the widths of interval-solutions are discussed. Section 6 concludes the paper.

2. THE INITIAL VALUE PROBLEM AND CLASSICAL RUNGE-KUTTA METHODS

The initial value problem consists in finding the function $y = y(t)$, such that

$$y' = f(t, y), \tag{1}$$

subject to an initial condition

$$y(t_0) = y_0 , \tag{2}$$

where $t \in [0, T]$, $y \in \Re^N$ and $f \in [0, T] \times \Re^N \to \Re^N$. We will assume that the solution of (1)-(2) exists and is unique (see e.g. [6]).

From the theory of the ordinary differential equations it is known that it takes place if the function f satisfies the Lipschitz condition, i.e. f is determined and continuous in the set $\{(t, y) : 0 \le t \le T, \ y \in \Re^N\}$ and there exists a constant $L > 0$ such that for each $t \in [0, T]$ and all $u, v \in \Re^N$ we have

$$\| f(t, u) - f(t, v) \| \le L \| u - v \| .$$

To carry out a single step by an m-stage Runge-Kutta method it is necessary to use the formula (see e.g.[7])

$$y_{k+1} = y_k + h \sum_{i=1}^{m} w_i \kappa_i , \qquad k = 0, 1, 2, \ldots, \tag{3}$$

where

$$\kappa_i = f\left(t_k + c_i h, \ y_k + h \sum_{j=1}^{m} a_{ij} \kappa_j \right), \qquad i = 1, 2, \ldots, m , \tag{4}$$

and

$$\sum_{i=1}^{m} w_i = 1, \qquad c_i = \sum_{j=1}^{m} a_{ij} . \tag{5}$$

The set of constant numbers w_i, c_i, a_{ij} characterize a particular method (these coefficients depend on the number of stages m and on the order p of the method).

The local truncation error of step $t_{k+1} = t_k + h$ for a Runge-Kutta method of order p can be written in the form (see e.g. [6] or [7])

$$\begin{aligned} r_{k+1}(h) &= \psi(t_k, y(t_k)) h^{p+1} + O(h^{p+2}) = \\ &= r_{k+1}^{(p+1)}(0) \frac{h^{p+1}}{(p+1)!} + r_{k+1}^{(p+2)}(\theta\, h) \frac{h^{p+2}}{(p+2)!} , \end{aligned} \tag{6}$$

where

$$0 < \theta < 1, \qquad \left| \frac{r_{k+1}^{(p+2)}(\theta h)}{(p+2)!} \right| \le M < \infty . \tag{7}$$

This error is equal to the difference between the exact value $y(t_k + h)$ and its approximation evaluated on the basis of the exact value $y(t_{k+1})$. The function $\psi(t, y(t))$ depends on coefficients w_i, c_i, a_{ij} and on partial derivatives of the function $f(t, y)$ in (1)-(2). The form of

$\psi(t, y(t))$ is very complicated and cannot be written in general form for an arbitrary order p (see e. g. [4], [6] or [7]).

3. IMPLICIT INTERVAL METHODS OF RUNGE-KUTTA TYPE

Let us denote:

Δ_t and Δ_y - sets in which the function $f(t, y)$ is defined, i.e.

$$\Delta_t = \{ t \in \Re : 0 \le t \le a \},$$

$$\Delta_y = \{ y = [y_1, y_2, \ldots, y_N]^T \in \Re^N : \underline{b_s} \le y_s \le \overline{b_s}, \quad s = 1, 2, \ldots, N \}.$$

$F(T, Y)$ - an interval extension of $f(t, y)$ (for a definition of interval extension see e.g. [1] or [8]),

$\Psi(T, Y)$ - an interval extension of $\psi(t, y)$ (staying in (6)).

Let us assume that:

- the function $F(T, Y)$ is defined and continues for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- the function $F(T, Y)$ is monotone with respect to inclusion, i. e.

$$T_1 \subset T_2 \wedge Y_1 \subset Y_2 \Rightarrow F(T_1, Y_1) \subset F(T_2, Y_2),$$

- if $d(A)$ denote the width of the interval then $\forall T \subset \Delta_t \wedge \forall Y \subset \Delta_y, \exists L > 0$ that

$$d(F(T, Y)) \le L(d(T) + d(Y)),$$

if $A = [A_1, A_2, \ldots, A_N]^T$, then $d(A) = \max\limits_{s=1,2,\ldots,N} d(A_s)$,

- the function $\Psi(T, Y)$ is defined for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- the function $\Psi(T, Y)$ is monotone with respect to inclusion.

For $t_0 = 0$ and $y_0 \in Y_0$ an implicit m-stage interval method of Runge-Kutta type is defined by the following formulas:

$$Y_n(t_0) = Y_n(0) = Y_0,$$

$$Y_n(t_{k+1}) = Y_n(t_k) + h \sum_{i=1}^{m} w_i K_{i,k}(h) + (\Psi(T_k, Y_n(t_k)) + [-\alpha, \alpha]) \cdot h^{p+1}, \tag{8}$$

$$k = 0, 1, \ldots, n-1,$$

where

$$K_{i,k}(h) = F\left(T_k + c_i h, \; Y_n(t_k) + h \sum_{j=1}^{m} a_{ij} K_{j,k}(h)\right), \qquad i = 1, 2, \ldots m, \tag{9}$$

$$\alpha = M h_0.$$

$M = [M_1, M_2, \ldots, M_N]^T$ is given by (7), and $p \le 2m$ for $m = 1,2,3$. In our opinion the implicit interval methods with a higher number of stages and with an order higher than four are not interesting on account of a great cost of calculations. Therefore, we consider the one-, two- and three-stage methods up to the fourth order inclusive.

If h_0 denotes a given number, then the step-size h of the method (8), where $0 < h \le h_0$, is calculated from the formula

$$h = \frac{\eta_m^*}{n}, \tag{10}$$

where

$$\eta_m^* = \min\{\eta_0, \eta_1, \ldots, \eta_m\}. \tag{11}$$

For $Y_0 \subset \Delta_y$ and $y_0 \in Y_0$ the numbers $\eta_1 > 0, \eta_2 > 0, \ldots, \eta_m > 0$ are evaluated in such a way that

$$Y_0 + \eta_i c_i F(\Delta_t, \Delta_y) \subset \Delta_y, \qquad i = 1, 2, \ldots, m, \tag{12}$$

and the number η_0 has to fulfil the condition

$$Y_0 + \eta_0 \sum_{i=1}^{m} w_i F(\Delta_t, \Delta_y) + (\Psi(\Delta_t, \Delta_y) + [-\alpha, \alpha\,]) \cdot h_0^p \subset \Delta_y. \tag{13}$$

We divide the interval $[0, \eta_m^*]$ into n parts by the points $t_k = kh,\ k = 0, 1, \ldots, n$, and the intervals T_k, which appear in the formulas (8)-(9), have to be taken in such a way that

$$t_k = kh \in T_k \subset [0, \eta_m^*].$$

For the methods given by (8)-(9), in each step of the method we have to solve a nonlinear interval system of equations of the form

where

$$T \subset \Delta_t \subset I(\Re),$$
$$X = (X_1, X_2, \ldots, X_N)^T \subset I(\Delta_t) \subset I(\Re^N),$$
$$G : I(\Delta_t) \times I(\Delta_y) \to I(\Re^N).$$

Assuming that G is a contracting mapping, from the fixed point theorem (see e.g. [9]) it follows that the iteration process

$$X^{(l+1)} = G(T, \quad X^{(l)}), \quad l \quad = 0, 1, \ldots, \tag{14}$$

is convergent to an element X^*, that is $\lim_{l \to \infty} X^l = X^*$, for an arbitrary choice of $X^{(0)} \subset I(\Delta_y)$. For system of equations (9) the iteration process (14) is of the form

$$K_{i,k}^{(l+1)}(h) = F\left(T_k + c_i \cdot h,\ Y_n(t_k) + h \cdot \sum_{j=1}^{m} a_{ij} \cdot K_{j,k}^{(l)}(h)\right), \tag{15}$$
$$i = 1, 2, \ldots, m, \qquad k = 0, 1, \ldots, n-1, \qquad l = 0, 1, \ldots,$$

where

$$K_{i,k}^{(0)}(h) = F(T_k + c_i h,\ Y_n(t_k)).$$

If in each equation of (15) we insert all approximations calculated so far, then we get the process of the form

$$K_{i,k}^{(l+1)}(h) = F\left(T_k + c_i \cdot h,\ Y_n(t_k) + h \cdot \left(\sum_{j=1}^{i-1} a_{ij} \cdot K_{j,k}^{(l+1)}(h) + \sum_{j=i}^{m} a_{ij} \cdot K_{j,k}^{(l)}(h)\right)\right),$$

and in this way we reduce the number of iterations.

In order to solve the initial value problem (1)-(2) by the implicit interval method of Runge-Kutta type we must determine the integration interval $[0, \eta^*_m]$ on which the method may be applied (see [10]).

4. THE EXACT REPRESENTATIONS OF THE NUMBERS IN THE COMPUTER

In numerical problems handled on the computer, one usually chooses the machine type which has a maximum number of significant digits. In the *Object Pascal* language this feature is represented by the ***Extended*** type. The value of a number w (in the ***Extended*** type) can be determined as follows (see [11] page 11-5)

$$w = \begin{cases} (-1)^s \, 2^{(c-16383)}(i.m) & \text{when} \quad 0 \le c < 2^{15} - 1, \\ (-1)^s \, Inf & \text{when} \quad c = 2^{15} \quad \text{and} \quad m = 0, \\ NaN & \text{when} \quad c = 2^{15} \quad \text{and} \quad m \ne 0, \end{cases}$$

where s is the sign bit (0 or 1), c denotes the exponent, m denotes the mantissa, and i denotes the digit (0 or 1) in front of the point in the mantissa (the numbers are normalized). If $c = 2^{15}$, then the number represented in this case is infinite (*Inf*) or is designated as *NaN* (not-a-number). An occurrence of *NaN* usually indicates an attempt to store a number with a magnitude which is larger than allowable (overflow) or an attempt to store a nonzero number with a magnitude which is smaller than allowable (underflow).

REMARK:

In the further considerations we do not deal with the cases *Inf* and *NaN*.

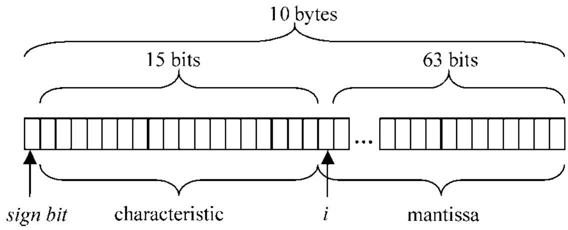

Fig. 1. Machine representation of a real number in the ***Extended*** type

The mantissa m of the number w in the ***Extended*** type occupies 64 bits and it can be shown that its value is $\sum_{i=0}^{63} \frac{m_i}{2^i}$, where $m_i = \{0, 1\}$.

Theorem 1.

Let A be a set of machine numbers (i. e. the numbers which have exact representations in the computer) and let operator $\circ$ *be an arbitrary element of the set* $\{+, -, \cdot\}$. *If quantities x, y belong to the set A, then the result of operation* $x \circ y = z$ *is determined in this set if and only if it is equal to its rounded value, i.e. when* $z = \mathrm{rd}(z)$.

Proof.

If z = rd(z) in any real type, then the cases NaN and Inf do not occur. The quantities x, y are exactly represented, and therefore x = rd(x) and y = rd(y). If an operation $\circ \in \{+, -, \cdot\}$ is exact, i. e. the result of this operation is not rounded, then

$$\underset{x,y\in A}{\forall}\ x \circ y = z \in A \Leftrightarrow z = \mathrm{fl}(x \circ y) = \mathrm{rd}(\mathrm{rd}(x) \circ \mathrm{rd}(y)) = \mathrm{rd}(x \circ y) = \mathrm{rd}(z).$$

The machine representation of a quantity z which is a result of the operation $\circ \in \{+, -, \cdot\}$ in any real type must have a finite expansion of the mantissa. So, in floating-point notation quantity z will occupy at most all bits of the mantissa in the real type considered (see also [12]).

We denote:

Z – a set of integer numbers,

N – a set of natural positive numbers,

n – the number of bits of the mantissa in any real type less 1.

The following quantities in any real type are exactly produced:

1) the integer numbers $A \in Z$,
2) the numbers of the form $\frac{A}{2^k}$, where $A \in Z$, $k \le n$, $k \in N$; for $k \le 0$ we get the case 1),

3) zero ($A = 0$),

4) the numbers of the form $\sum_{i=0}^{n} \frac{A_i}{2^i}$, where $A_i \in \mathrm{Z}$, because $\sum_{i=0}^{n} \frac{A_i}{2^i} = \frac{1}{2^n} \sum_{i=1}^{n} A_i 2^{n-i}$, where $n - i \geq 0$, and hence $2^{n-i} \in \mathrm{Z}$, which means we get the numbers of the form given in 2),

5) the numbers of the form $B + \sum_{i=0}^{n} \frac{A_i}{2^i}$, where $A_i, B \in Z$,

because, $B + \sum_{i=0}^{n} \frac{A_i}{2^i} = \frac{B}{2^0} + \sum_{i=0}^{n} \frac{A_i}{2^i} = \sum_{i=0}^{n} \frac{C_i}{2^i}$ where $C_i \in Z$ and

$C_i = \begin{cases} A_0 + B, & \text{for } i = 0, \\ A_i, & \text{for } i > 0, \end{cases}$ which means we get the numbers of the form given in 4).

So, in order to show that a number is represented exactly, one should verify that it is possible to represent it in the form l), 2) or 3).

REMARK:

For the exact numbers x, y, z, and $\circ \in \{+, -, \cdot\}$, an operation $x \circ y = z$ is commutative, i. e. $x \circ y = y \circ x = z$.

In the further part of this paper the notion *exact number* will denote t*he exact machine number*.

In Table 1 we present the results of the operations from the set {+, -, } using exact quantities, where $A, B, C \in Z$, $k, l, m \in N$ and $k, l, m \leq n$.

Table 1. The operations +, -, with exact quantities

X	0			$\in Z$		$\frac{A}{2^k}$
Y	0	$\in Z$	$\frac{B}{2^l}$	$\in Z$	$\frac{B}{2^l}$	$\frac{B}{2^l}$
$X \pm Y$	0	$\in \mathrm{Z}$	$\pm \frac{B}{2^l}$	$\in Z$	$\frac{C}{2^m}$	$\frac{C}{2^m}$
$X \cdot Y$	0			$\in Z$	$\frac{C}{2^m}$	$\frac{C}{2^m}$

From Table 1 it is obvious that the presented results have exact machine representations.

5. THE MINIMIZATION OF THE WIDTHS OF INTERVAL SOLUTIONS

5.1. One-stage implicit interval methods

Theorem 2.

In floating-point interval arithmetic the coefficients of the one-stage implicit method of the second order which are represented exactly.

Proof.

The coefficients of the one-stage method of the second order, called the midpoint method, are as follows: $w_1 = 1$, $c_1 = \frac{1}{2}$, $a_{11} = \frac{1}{2}$. In the computer all these values are represented exactly.

5.2. Two-stage implicit interval methods

Theorem 3.

In floating-point interval arithmetic there are two-stage methods of third- and fourth-order which have coefficients with exact representation in the computer.

Proof.

For each of the methods we must carry out the proof separately (see also [13]).

Two-stage methods of the third order ($m = 2$, $p = 3$) - general solution

All families of solutions without irrational values contain the following formula:

$$c_i = \frac{1}{2} - \frac{1}{12c_j - 6} \text{ for } c_j \neq \frac{1}{2}, \text{ where } i, \underset{i \neq j}{j =} 1, 2 .$$

If for exact values of c_j there exist exact values of c_i ($i, \underset{i \neq j}{j =} 1, 2$), then (for $A, A_1 \in Z$ and $k, k_1 \in N$) we should consider the following cases:

c_j	0			$\in Z$			$\frac{A_1}{2^{k_1}}$		
c_i	0	$\in Z$	$\frac{A}{2^k}$	0	$\in Z$	$\frac{A}{2^k}$	0	$\in Z$	$\frac{A}{2^k}$

1) $c_j = 0$, then $c_i = \frac{2}{3}$ $\left(\neq 0, \notin Z, \neq \frac{A}{2^k}\right)$,

2) $c_j \in Z$ or $c_j = \frac{A}{2^k}$,

2.1) $c_i = 0$, then $\frac{1}{2} - \frac{1}{12c_j - 6} = 0 \quad \Rightarrow \quad c_j = \frac{2}{3} \quad \left(\notin Z, \neq \frac{A}{2^k} \right)$,

2.2) $c_i \in Z$, then $1 - \frac{1}{6c_j - 3} = 2c_i \quad \Rightarrow \quad \frac{1}{6c_j - 3} = 1 - 2c_i \quad (\in Z)$

for $c_j \in Z$: $\frac{1}{6c_j - 3} \notin Z$,

for $c_j = \frac{A}{2^k}$: $\frac{1}{6c_j - 3} = \frac{2^{k-1}}{3(A - 2^{k-1})} \notin Z$, (numerator is even, and denominator is the multiplicity of number 3),

2.3) $c_i = \frac{A_1}{2^{k_1}}$, then $c_i = \frac{1}{2} - \frac{1}{12c_j - 6} = \frac{A_1}{2^{k_1}} \Rightarrow \quad \frac{1}{12c_j - 6} \quad \left(\in Z, = \frac{A_2}{2^{k_2}} \right)$,

for $c_j \in Z$: $\frac{1}{12c_j - 6} \left(\notin Z, \neq \frac{A_2}{2^{k_2}} \right)$,

for $c_j = \frac{A}{2^k}$: $\frac{1}{12c_j - 6} = \frac{2^{k-1}}{3(A - 2^{k-1})} \left(\notin Z, \neq \frac{A_2}{2^{k_2}} \right)$.

Two-stage methods of the third order ($m = 2$, $p = 3$) - semi-implicit methods

All families of solutions without irrational values contain one of the following formulas:

a) $c_i = \frac{1}{2} - \frac{1}{12c_j - 6}$ for $c_j \neq \frac{1}{2}$,

b) $c_i = \frac{1}{2} + \frac{1}{12c_j - 6}$ for $c_j \neq \frac{1}{2}$,

c) $c_i = \frac{1}{2} - \frac{1}{6c_j}$ for $c_j \neq 0$, where $i, j \underset{i \neq j}{=} 1, 2$.

REMARKS:

- Point a) has been considered for the general solution of these methods ($m = 2$, $p = 3$).
- Point b) follows from point a):

$$c_i = \frac{1}{2} + \frac{1}{12c_j - 6} = \frac{1}{2} - \frac{-1}{12c_j - 6}, \text{ for } c_j \neq \frac{1}{2}.$$

This formula differs from the formula given in point a) only by the constant equals -1 in the numerator of the second fraction (it does not essentially modify the proof).

- In point c) we investigate whether for the exact values of $c_j \neq 0$ there exist exact values of c_i $(i, j \underset{i \neq j}{=} 1, 2)$, where $c_i = \frac{1}{2} - \frac{1}{6c_j}$.

Hence, for $A, A_1 \in Z$ and $k, k_1 \in N$ we should consider the following cases:

c_i	0		$\in Z$		$\frac{A_1}{2^{k_1}}$	
c_j	$\in Z$	$\frac{A}{2^k}$	$\in Z$	$\frac{A}{2^k}$	$\in Z$	$\frac{A}{2^k}$

1) $c_i = 0$, then $\frac{1}{2} - \frac{1}{6c_j} = 0 \quad \Rightarrow \quad c_j = \frac{2}{3} \quad \left(\notin Z, \neq \frac{A}{2^k}\right)$,

2) $c_i \in Z$, then $1 - \frac{1}{3c_j} = 2c_i \quad \Rightarrow \quad \frac{1}{3c_j} = 1 - 2c_i \quad (\in Z)$

for $c_j \in Z$: $\frac{1}{3c_j} \notin Z$,

for $c_j = \frac{A}{2^k}$: $\frac{1}{3c_j} = \frac{2^k}{3A} \notin Z$, (the numerator is even and the denominator is the multiplicity of number 3),

$c_i = \frac{A_1}{2^{k_1}}$, then $c_i = \frac{1}{2} - \frac{1}{6c_j} = \frac{A_1}{2^{k_1}} \quad \Rightarrow \quad \frac{1}{6c_j} \quad \left(\in Z, = \frac{A_2}{2^{k_2}}\right)$

for $c_j \in Z$ we get $\frac{1}{6c_j}\left(\notin Z, \neq \frac{A_2}{2^{k_2}}\right)$,

for $c_j = \frac{A}{2^k}$ we obtain $\frac{1}{6c_j} = \frac{2^k}{3A}\left(\notin Z, \neq \frac{A_2}{2^{k_2}}\right)$.

Two-stage methods of the third order ($m = 2, \ p = 3$) - symmetrical methods

All solutions contain irrational values.

Two-stage methods of the fourth order ($m = 2, \ p = 4$) - general solution

All solutions contain irrational values.

5.3. Three-stage implicit interval methods

Theorem 4.

In floating-point interval arithmetic there are no three-stage methods of the fourth order with coefficients that are represented exactly in the computer.

Proof.

For each of the methods we should carry out the proof separately (see also [13]).

Three-stage methods of the fourth order ($m = 3,\ p = 4$) - general solution

All families of solutions contain the following formula:

$$w_2 = \frac{\frac{1}{2} - c_3 - w_1(c_1 + c_2)}{c_2 - c_3},$$

where $w_1 = \dfrac{c_2c_3 - \frac{1}{2}(c_2 + c_3) + \frac{1}{3}}{(c_1 - c_2)(c_1 - c_3)}$. Hence, if $w_2 \neq 0$, $c_i \neq c_j$ for L, $j = 1, 2, 3$, then

$$w_2 = \frac{c_1c_3 - \frac{1}{2}(c_1 + c_3) + \frac{1}{3}}{(c_1 - c_2)(c_3 - c_2)}.$$

If for the exact values of c_1, c_2 and c_3 ($c_i \neq c_j$ for $i, j = 1, 2, 3$) there exist the exact value of w_2 ($w_2 \neq 0$), then (for $A, A_1 \in Z$ and $k, k_1 \in N$ $(k, k_1 > 0)$) we should consider twelve cases presented in the following table:

<table>
<tr><td>c_1</td><td colspan="2">0</td><td>$\in Z$</td><td>$\frac{A}{2^k}$</td><td>$\in Z$</td><td>$\frac{A}{2^k}$</td></tr>
<tr><td>c_2</td><td>$\in Z$</td><td>$\frac{A}{2^k}$</td><td colspan="2">0</td><td>$\frac{A}{2^k}$</td><td>$\in Z$</td></tr>
<tr><td>c_3</td><td>$\frac{A}{2^k}$</td><td>$\in Z$</td><td>$\frac{A}{2^k}$</td><td>$\in Z$</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">w_2</td><td colspan="6">$\in Z$</td></tr>
<tr><td colspan="6">$\frac{A_1}{2^{k_1}}$</td></tr>
</table>

1) $c_1 = 0$, then $w_2 = \dfrac{\frac{1}{2}c_3 - \frac{1}{3}}{c_2(c_3 - c_2)} = \dfrac{3c_3 - 2}{6c_2(c_3 - c_2)}$,

where $\left(c_2 \in Z,\ c_3 = \dfrac{A}{2^k}\right)$ or $\left(c_2 = \dfrac{A}{2^k},\ c_3 \in Z\right)$,

2) $c_2 = 0$, then $w_2 = 1 + \dfrac{\frac{1}{3} - \frac{1}{2}(c_1 + c_3)}{c_1c_3} = 1 + \dfrac{1}{3c_1c_3} - \dfrac{1}{2}\left(\dfrac{1}{c_1} + \dfrac{1}{c_3}\right)$,

where $\left(c_1 \in Z,\ c_3 = \dfrac{A}{2^k}\right)$ or $\left(c_1 = \dfrac{A}{2^k},\ c_3 \in Z\right)$,

3) $c_3 = 0$, then $w_2 = \dfrac{\frac{1}{2}c_1 - \frac{1}{3}}{c_2(c_1 - c_2)} = \dfrac{3c_1 - 2}{6c_2(c_1 - c_2)}$,

where $\left(c_1 \in Z,\, c_2 = \frac{A}{2^k}\right)$ or $\left(c_1 = \frac{A}{2^k},\, c_2 \in Z\right)$.

The case in 3) is analogous to the case presented in 1). It means it is enough to prove the cases in 1) and 2).

1) $c_1 = 0 \quad \Rightarrow \quad w_2 = \dfrac{3c_3 - 2}{6c_2(c_3 - c_2)}$

1.1) $w_2 \in Z$

a) $c_2 \in Z,\, c_3 = \frac{A}{2^k}$, then $w_2 = \dfrac{\frac{3A}{2^k} - 2}{6c_2\left(\frac{A}{2^k} - c_2\right)} = \dfrac{3A - 2^{k+1}}{6c_2\left(A - 2^k c_2\right)}$,

b) $c_2 = \frac{A}{2^k},\, c_3 \in Z$, then $w_2 = \dfrac{3c_3 - 2}{6\frac{A}{2^k}\left(c_3 - \frac{A}{2^k}\right)} = \dfrac{3 \cdot 2^{2k-1}c_3 - 2^{2k}}{3A\left(2^k c_3 - A\right)}$,

for $A \in \mathrm{Z}, \quad k \in \mathrm{N}\ (k > 0) \Rightarrow 2k - 1 > 0$.

In both cases the denominator is divisible by 3, but the second element of the numerator is even. Thus, the numerator is not the multiplicity of the denominator and $w_2 \notin Z$.

1.2) $w_2 = \dfrac{A_1}{2^{k_1}}$

a) $c_2 \in Z,\, c_3 = \frac{A}{2^k}$, then $w_2 = \dfrac{\frac{3A}{2^k} - 2}{6c_2\left(\frac{A}{2^k} - c_2\right)} = \dfrac{3A - 2^{k+1}}{6c_2\left(A - 2^k c_2\right)}$,

b) $c_2 = \frac{A}{2^k},\, c_3 \in \mathrm{Z}$, then $w_2 = \dfrac{3c_3 - 2}{6\frac{A}{2^k}\left(c_3 - \frac{A}{2^k}\right)} = \dfrac{3 \cdot 2^{2k-1}c_3 - 2^{2k}}{3A\left(2^k c_3 - A\right)}$,

for $A, A_1 \in \mathrm{Z}, \quad k, k_1 \in \mathrm{N}\ (k, k_1 > 0) \Rightarrow 2k - 1 > 0$.

If w_2 is of the form $\frac{A_1}{2^{k_1}}$, then the numerator must be the multiplicity of number 3. But the second element of the numerator is even. Thus, the coefficient w_2 cannot be represented in the form $\frac{A_1}{2^{k_1}}$.

2) $c_2 = 0 \quad \Rightarrow \quad w_2 = 1 + \dfrac{1}{3c_1c_3} - \dfrac{1}{2}\left(\dfrac{1}{c_1} + \dfrac{1}{c_3}\right)$

2.1) $w_2 \in \mathrm{Z}$

a) $c_1 \in \mathrm{Z},\, c_3 = \frac{A}{2^k}$,

b) $c_1 = \frac{A}{2^k}$, $c_3 \in Z$.

In both cases $w_2 = 1 + \frac{2^{k+1} - 3\left(A + 2^k c_i\right)}{6 c_i A}$, where $i = 1, 3$, $A \in Z$, and $k \in N$ $(k > 0)$.

If $w_2 \in Z$, then the numerator must be divisible by 3. But the first element of the numerator is even. It means that the numerator is not the multiplicity of the denominator and $w_2 \notin Z$.

2.2) $w_2 = \frac{A_1}{2^{k_1}}$

a) $c_1 \in Z$, $c_3 = \frac{A}{2^k}$,

b) $c_1 = \frac{A}{2^k}$, $c_3 \in Z$.

In both cases $w_2 = 1 + \frac{2^{k+1} - 3\left(A + 2^k c_i\right)}{6 c_i A}$, where $i = 1, 3$, $A, A_1 \in Z$, and $k, k_1 \in N$ $(k, k_1 > 0)$.

If w_2 is of the form $\frac{A_1}{2^{k_1}}$, then the numerator of the expression $\frac{2^{k+1} - 3\left(A + 2^k c_i\right)}{6 c_i A}$ must be the multiplicity of number 3. This is impossible, because the first element of the numerator is even. Thus, the coefficient w_2 can not be represented in the form $\frac{A_1}{2^{k_1}}$.

Three-stage methods of the fourth order ($m = 3$, $p = 4$) - general solution with the supplementary conditions

All solutions contain irrational values.

Three-stage methods of the fourth order ($m = 3$, $p = 4$) - semi-implicit methods

All families of solutions contain the following formula:

$$w_3 = \frac{c_1 c_2 - \frac{1}{2}\left(c_1 + c_2\right) + \frac{1}{3}}{\left(c_1 - c_3\right)\left(c_2 - c_3\right)},$$

where $w_3 \neq 0$, $c_i \neq c_j$ for $i, j = 1, 2, 3$.

This formula is similar to the case considered for the general solution.

Three-stage methods of the fourth order ($m = 3$, $p = 4$) - semi-implicit diagonal methods

All solutions contain irrational values.

Three-stage methods of the fourth order ($m = 3,\ p = 4$) - symmetrical methods

All families of solutions contain the formula as follows:

$$w_2 = 1 - \frac{1}{3(2c_i - 1)^2},$$

for $i = 1, 2, 3$ and $c_i \neq \frac{1}{2}$.

If for the exact value of c_i there exist the exact value of w_2, then it is necessary to consider the following cases:

c_i	0			$\in Z$			$\frac{A_1}{2^{k_1}}$		
w_2	0	$\in Z$	$\frac{A}{2^k}$	0	$\in Z$	$\frac{A}{2^k}$	0	$\in Z$	$\frac{A}{2^k}$

where $A, A_1 \in Z$ and $k, k_1 \in N$.

3) $c_i = 0$, then $w_2 = \frac{2}{3} \quad \left(\neq 0, \notin Z, \neq \frac{A}{2^k}\right)$,

4) $c_i \in Z$ or $c_i = \frac{A}{2^k}$,

4.1) $w_2 = 0$, then $1 - \frac{1}{3(2c_i - 1)^2} = 0 \quad \Rightarrow \quad c_i = \frac{1}{2} + \frac{\sqrt{3}}{6} \quad \left(\notin Z, \neq \frac{A}{2^k}\right)$,

4.2) $w_2 \in Z$, then $\frac{1}{3(2c_i - 1)^2} \quad (\in Z)$,

for $c_i \in Z$ we get $\frac{1}{3(2c_i - 1)^2} \notin Z$,

for $c_i = \frac{A_1}{2^{k_1}}$ we obtain $\frac{1}{3(2c_i - 1)^2} = \frac{2^{2k-2}}{3\left(A - 2^{k-1}\right)^2} \notin Z$ (the numerator is even and the denominator is the multiplicity of number 3),

4.3) $w_2 = \frac{A}{2^k}$, then $w_2 = 1 - \frac{1}{3(2c_i - 1)^2} = \frac{A_1}{2^{k_1}} \quad \Rightarrow \quad \frac{1}{3(2c_i - 1)^2} \quad \left(= \frac{A_2}{2^{k_2}}\right)$,

for $c_i \in Z$ we receive $\frac{1}{3(2c_i - 1)^2} \neq \frac{A_2}{2^{k_2}}$,

for $c_i = \frac{A_1}{2^{k_1}}$ we have $\frac{1}{3(2c_i - 1)^2} = \frac{2^{2k_1 - 2}}{3(2A_1 - 1)^2} \neq \frac{A_2}{2^{k_2}}$.

6. REMARKS

In this paper it has been shown that there is only one implicit interval method of Runge-Kutta type (up to the fourth order inclusive) which has coefficients represented exactly in the computer. It is the one-stage method of the second order called the midpoint rule. For the other methods there is no combination of coefficients for which the widths of interval-solutions are minimal.

We tried to prove that it was possible to minimize the width of interval solutions with respect to some constants occurring in the estimation of this width (see [1] and [2]). Unfortunately, it appears to be impossible. Another minimization, i.e. the minimization with respect of these coefficients for the function $\psi(t_k, y(t_k))$, see (6), is considered in the theory of classical Runge-Kutta methods (see e.g. [6]). In the case of interval methods of this type we can do the same. From our experience it follows that this is the only way in which the diameters of interval solutions are minimized.

References

[1] A. Marciniak and B. Szyszka, *One- and Two-Stage Implicit Interval Methods of Runge-Kutta Type,* Computational Methods in Science and Technology 5, OWN, Poznań 1999.

[2] A. Marciniak, K. Gajda, and B. Szyszka, Three- *and Four-Stage Implicit Interval Methodss of Runge-Kutta Type,* Computational Methods in Science and Technology 6, OWN, Poznań 2000.

[3] S. A. Kalmykov, Yu. I. Šokin, and Z. H. Yuldašew, *Methodss of Interval Analysis* [in Russian], Nauka, Novosibirsk 1986.

[4] A. Marciniak., K. Gajda, A. Marlewski, and B. Szyszka, *The Concept of Object-Oriented System for Solving the Initial Value Problem by Interval Methods of Runge-Kutta Type* [in Polish], Pro Dialog 8, Nakom, Poznań 1999, 39-62.

[5] A. Marciniak, K. Gajda, and B. Szyszka, *Interval Methods of Runge-Kutta Type in Floating-Point Interval Arithmetics* [in Polish], Technical Report RB-028/2000, Poznań University of Technology, Institute of Computing Science, Poznań 2000.

[6] A. Krupowicz, *Numerical Methods of Initial Value Problems of Ordinary Differential Equations* [in Polish], PWN, Warszawa 1986.

[7] J. C. Butcher, *The Numerical Analysis of Ordinary Differential Equations. Runge-Kutta and General Linear Methods,* J. Wiley&Sons, Chichester 1987.

[8] Yu. I. Šokin, *Interval Analysis,* [in Russian], Nauka, Novosibirsk 1981.

[9] J. Kudrewicz., *Analizafunkcjonalna dla automatyków i elektroników,* Państwowe Wydawnictwo Naukowe, Warszawa 1976.

[10] A. Marciniak, *Finding the Integration Interval for Interval Methods of Runge-Kutta Type in Floating-PointInterval Arithmetics,* Pro Dialog 10, Nakom, Poznań 2000, 35-45.

[11] *BorlandDelphi 6. Object Pascal. Language Guide,* Borland, Scotts Valley 2001.

[12] A. Marciniak and A. Marlewski, *Interval Representation of Non-Machine Numbers in Object Pascal* [in Polish], Pro Dialog 7, Nakom, Poznań 1998, 75-100.

[13] A. Marciniak and B. Szyszka, *Interval Metods of Runge-Kutta Type in Floating-Point Interval Arithmetics* [in Polish], Technical Report RB-022/2001, Poznań University of Technology, Institute of Computing Science, Poznań 2001.