

## TWO INTERACTING PARTICLES IN A PARABOLIC WELL: HARMONIUM AND RELATED SYSTEMS\*

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Abstract: The quasi-exactly solvable problem of two interacting electrons confined by a parabolic potential (harmonium) has been generalized for the case of two arbitrary particles. Several new features of the analytical solutions are presented.

### 1. INTRODUCTION

Many quantum-chemical methods have been derived in a more or less straightforward way from analytically solvable models such as the hydrogen-like atom or the harmonic oscillator. Analytically solvable models supply also invaluable opportunities for checking correctness and ranges of applicability of a variety of approximations. During the last decade several new quasi-analytically solvable models have been introduced to quantum chemistry and to atomic and molecular physics. Probably the most interesting and the best known is the model describing two interacting electrons confined in a harmonic oscillator potential. It was applied to numerical studies on electron correlation by Kestner and Sinanoğlu already in the sixties [1]. The analytic solutions have been obtained (independently of the earlier works of Singh [2] and Znoil [3]) by Taut [4]. Later this system, referred to as *harmonium*, has been used by Taut *et al.* as a reference model in studies on density functional theory [5, 6] and by Cioslowski and Pernal on some aspects of the electron correlation [7]. The properties of the energy spectrum of harmonium have been recently analyzed by the present authors [8, 9]. Some of its features, just discovered, are reported in this paper. Similar models have also been used to study properties of quantum dots [10-14]. In this context one should mention works by Ghosh and Samantha [15, 16] who added an extra linear term to the interaction potential and, in effect, got one additional coupling constant in the Hamiltonian. This trick leads to a larger set of analytical solutions of the corresponding eigenvalue problem. Generalizations of this approach to more than two particles constitute a difficult challenge but are by no means hopeless [17, 18].

The two-particle Schrödinger equation describing harmonium as well as its modifications and generalizations may be separated to two spherically-symmetric one-particle equations, one of them describing the motion of the center of mass in the confining potential and the other one describing the relative motion of the particles [4]. The center of mass equation, in the case of a parabolic confinement, reduces to a three-dimensional harmonic oscillator eigenvalue

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\* Dedicated to the memory of Professor Jacek Rychlewski

problem and is easily solvable. In the equation describing the relative motion, the angular part of the wavefunction may be separated and expressed in terms of spherical harmonics. Analytical solutions of the remaining radial equation were obtained, for a discrete set of the coupling constants defining the effective potential, in the case of the pure Coulomb interaction by Taut [4] and in the case when an additional linear term appears by Ghosh and Samantha [15, 16]. General methods of solving similar equations, for a wide class of potentials, were formulated by Singh *et al.* [2] and by Znoil [3] a quarter of a century ago and recently extended by Bose and Gupta [19]. However, these mathematically oriented works have been noticed by neither quantum chemists nor solid state physicists. In the present paper we generalize the model of Taut [4] by introducing an additional coupling constant which describes the strength of the interaction. Also the method of solving the equation has been formulated in a general form. The recurrent relations which determine the wavefunctions have been transformed to an easily solvable eigenvalue problem of a symmetric tridiagonal matrix. In effect several new features of the analytical solutions have been noticed. The inspiration for this work has been taken from a general approach described in Refs. [2, 3, 19]. Consequently, the method presented in this paper is also an illustration of a general approach which may be used to construct analytically solvable models for a large number of interaction potentials.

## 2. GENERAL FORMULATION

The six-dimensional Schrödinger eigenvalue problem describing two particles with masses  $m_1$  and  $m_2$ , interacting by a potential  $V(|r_1 - r_2|)$  and confined by a harmonic-oscillator-type potential, i.e.

$$\left[ \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{\omega^2}{2} (m_1 r_1^2 + m_2 r_2^2) + V(|r_1 - r_2|) \right] \Psi(r_1, r_2) = E\Psi(r_1, r_2), \quad (1)$$

upon the substitution

$$\mathbf{R} = (m_1 r_1 + m_2 r_2) / (m_1 + m_2), \quad \text{and} \quad \mathbf{r} = r_1 - r_2, \quad (2)$$

may be separated to two spherically-symmetric one-particle equations:

$$\left[ -\frac{1}{2M} \nabla_R^2 + \frac{M}{2} \omega^2 R^2 \right] \Xi_{\nu\lambda\mu}(\mathbf{R}) = E_{\nu\lambda}^R \Xi_{\nu\lambda\mu}(\mathbf{R}), \quad (3)$$

and

$$\left[ -\frac{1}{2m} \nabla_r^2 + \frac{m}{2} \omega^2 r^2 + V(r) \right] \Phi_{nlm}(\mathbf{r}) = E_{nl}^r \Phi_{nlm}(\mathbf{r}), \quad (4)$$

where

$$\Psi(r_1, r_2) = \Xi_{\nu\lambda\mu}(\mathbf{R}) \Phi_{nlm}(\mathbf{r}), \quad E = E_{\nu\lambda}^R + E_{nl}^r, \quad (5)$$

$M = m_1 + m_2$  and  $m = m_1 m_2 / M$ .

The center of mass equation, (3), corresponds to a three-dimensional harmonic oscillator of the mass  $M$  and its solutions may be found in all textbooks of quantum mechanics. Equation (4) describes the relative motion. Due to its spherical symmetry

$$\Phi_{nlm}(\mathbf{r}) = \frac{1}{r} \phi_{nl}(r) Y_{lm}(\hat{r}), \quad (6)$$

where  $l, m$  are the angular momentum quantum numbers,  $Y_{lm}(\hat{r})$  are spherical harmonics, and  $n = 1, 2, \dots$  numbers consecutive energies for a given  $l$ . If we set, for simplicity,  $m_1 = m_2 = 1$  then  $m = 1/2$  and the radial equation reads

$$\left[ -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{1}{4} \omega_2 r^2 + V(r) \right] \phi_{nl}(r) = E_{nl}^r \phi_{nl}(r). \quad (7)$$

Equation (7) may be solved numerically, to practically unlimited accuracy, for all potentials  $V$  which are sufficiently regular and for which bound-state solutions exist. However, for a number of potentials  $V$  it may also be solved analytically. In this work we limit the discussion to the case of Coulomb-type potentials.

### 3. THE COULOMB POTENTIAL

If  $V(r) = 1/r$ , the properties of Eq. (7) were discussed in detail in the literature [1, 4-7] and the corresponding system is referred to as harmonium. By taking  $V(r) = \xi/r$  one may extend the range of systems described by this model. For example,  $\xi = -1$  may correspond to a confined hydrogen atom or muonic atom (if  $m_1 \neq m_2$ ) or to a confined positronium (if  $m_1 = m_2 = 1$ ) [9]. Besides, by adding an additional coupling constant we can get a deeper insight into the structure of the analytical solutions.

Upon the substitution

$$\rho = \sqrt{\omega} r, \quad (8)$$

Equation (7), with  $V = \xi/r$ , becomes

$$\left[ -\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2} + \frac{1}{4} \rho^2 + \frac{s}{\rho} \right] \phi_{nl}^s(\rho) = \varepsilon_{nl} \phi_{nl}^s(\rho) \quad (9)$$

where

$$s = \frac{\xi}{\sqrt{\omega}}, \quad \text{and} \quad \varepsilon_{nl} = E_{nl}^r / \omega. \quad (10)$$

In this section we present a general method of finding the analytical solutions of Eq. (9). The method has two advantages: it may be easily generalized for a large set of the interaction potentials  $V$  and it utilizes only standard computational techniques of linear algebra.

The asymptotically correct solutions of Eq. (9) have the form

$$\phi_{nl}^s(\rho) = \rho^{l+1} e^{-\rho^{2/4}} P_{nl}^{(sp)}(\rho), \quad (11)$$

where  $P_{nl}^{(sp)}(0)$  has to be finite and

$$\lim_{\rho \rightarrow \infty} \phi_{nl}^s(\rho) = 0 \quad (12)$$

We are looking for the analytical solutions (11) with

$$P_{nl}^{(sp)}(\rho) = \sum_{i=0}^p a_i \rho^i. \quad (13)$$

By substitution of Eqs. (11) and (13) to Eq. (9) we get

$$\begin{aligned} B_0 a_0 + C_1 a_1 &= 0, \\ A_i a_i + B_{i+1} a_{i+1} + C_{i+2} a_{i+2} &= 0, \quad i = 0, 1, 2, \dots, p-1 \end{aligned} \quad (14)$$

where

$$\begin{aligned} A_i &= \varepsilon_{nl} - i - l - 3/2 \\ B_i &= -s \\ C_i &= i(i + 2l + 1). \end{aligned} \quad (15)$$

If  $a_p \neq 0$  but  $a_{p+1} = a_{p+2} = \dots = 0$  then, according to the recurrence relations (14),  $A_p = 0$ , i.e.

$$\varepsilon_{nl} = p + l + 3/2 \quad (16)$$

and, consequently,

$$A_i = \omega(p - i). \quad (17)$$

Non-trivial solutions of Eqs. (14) exist if

$$W_p = \begin{vmatrix} B_0 & C_1 & 0 & \dots & 0 & 0 & 0 \\ A_0 & B_1 & C_2 & \dots & 0 & 0 & 0 \\ 0 & A_1 & B_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_{p-2} & B_{p-1} & C_p \\ 0 & 0 & 0 & \dots & 0 & A_{p-1} & B_p \end{vmatrix} = 0 \quad (18)$$

The determinant fulfills recurrent relations

$$\begin{aligned} W_0 &= B_0, \\ W_1 &= B_1 W_0 - A_0 C_1, \\ W_i &= B_i W_{i-1} - A_{i-1} C_i W_{i-2}, \quad i = 2, 3, \dots, p \end{aligned} \quad (19)$$

which make its evaluation very simple.

The determinant  $W_p$  depends on the indices  $i, p, l$  and on the parameter  $s$ . Therefore, for given  $l$  and  $p$ , Eq. (18) determines a discrete set  $s = s(l, p)$ ,  $i = 0, 1, 2, \dots, p$  for which the analytical solutions of Eq. (9) exist. The energies corresponding to the analytical solutions may be expressed as

$$E_{nl}^r = \omega_i(l, p)(p + l + 3/2),$$

where  $\omega_i(l, p) = \xi^2/s_i(l, p)^2$  and  $n$  is a function of  $p$  and  $i$ .

#### 4. TRANSFORMATION OF THE RECURRENT RELATIONS

It is convenient to transform Eqs. (14) to an eigenvalue problem of a symmetric matrix. Let

$$\alpha_i = a_i/q_i. \tag{20}$$

Then Eqs. (14) may be rewritten as

$$\begin{aligned} \tilde{B}_0\alpha_0 + \tilde{C}_1\alpha_1 &= 0 \\ \tilde{A}_i\alpha_i + \tilde{B}_{i+1}\alpha_{i+1} + \tilde{C}_{i+2}\alpha_{i+2} &= 0, \quad i = 0, 1, 2, \dots, p-1, \end{aligned} \tag{21}$$

where  $\tilde{A}_i = p_{i+1}A_iq_i$ ,  $\tilde{B}_i = p_iB_iq_i$  and  $\tilde{C}_{i+1} = p_iC_{i+1}q_{i+1}$ . As one can easily see, if we set

$$p_0 = 1, \quad p_i = \sqrt{\frac{C_i}{A_{i-1}} \frac{B_{i-1}}{B_i}} p_{i-1} \tag{22}$$

and

$$q_0 = 1, \quad q_i = \sqrt{\frac{A_{i-1}}{C_i} \frac{B_{i-1}}{B_i}} q_{i-1} \tag{23}$$

then

$$\tilde{A}_i = \tilde{C}_{i+1} \equiv D_{i+1} = B_0 \sqrt{\frac{C_{i+1}A_i}{B_{i+1}B_i}}, \tag{24}$$

and

$$\tilde{B}_{i+1} = B_0, \tag{25}$$

where we used the relation

$$p_iq_i = \frac{B_0}{B_i} p_0q_0. \tag{26}$$

Consequently, Eq. (18) may be transformed to an eigenvalue problem of a symmetric tridiagonal matrix

$$(T + B_0I)\alpha = 0, \tag{27}$$

where  $I$  is a unit matrix,  $-B_0$  stands for the eigenvalue,

$$T_{ik} = D_{i+1}\delta_{k,i+1} + D_{k+1}\delta_{i,k+1}, \quad i, k = 0, 2, \dots, p, \quad (28)$$

and  $\alpha = \{\alpha_0, \alpha_1, \dots, \alpha_p\}$ . The secular determinant is related to  $W$  as

$$\det|T + B_0 I| = C^{-1}W_p, \quad (29)$$

where

$$C = \prod_{i=0}^p p_i q_i = \prod_{i=0}^p \frac{B_0}{B_i}. \quad (30)$$

## 5. THE ANALYTICAL SOLUTIONS

The procedure described in the previous section is quite general and may be applied to an arbitrary three-term recurrent relation. If the confined particles are described by Eq. (9) then, according to Eqs. (15), (17) and (24),

$$B_0 = -s, \quad D_i = \sqrt{i(p-i+1)(i+2l+1)}, \quad (31)$$

and the secular equation may be written as

$$\det|T - sI| = \begin{vmatrix} -s & D_1 & 0 & \dots & 0 & 0 \\ D_1 & -s & D_2 & \dots & 0 & 0 \\ 0 & D_2 & -s & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -s & D_p \\ 0 & 0 & 0 & \dots & D_p & -s \end{vmatrix} = 0. \quad (32)$$

The eigenvalues  $s$  determine the conditions for the coupling constants  $\xi$  and  $\omega$  under which  $\phi_{ni}^s(\rho)$  given by Eq. (11) is a solution of Eq. (9). Since  $p$  may be an arbitrary non-negative finite integer, the discrete set of values of  $i$  for which Eq. (9) may be solved analytically is arbitrarily large, though finite. Solutions of Eq. (32) and the corresponding energies, for  $l = 0$  and  $p \leq 11$ , are collected in Table 1.

The sign of  $\xi$  is the same as the sign of  $s$  [cf. Eq. (10)]. Therefore  $s < 0$  corresponds to the confined positronium and  $s > 0$  corresponds to harmonium. The case of  $s = 0$  corresponds to the spherical harmonic oscillator. The spectrum of  $T$  is symmetric, i.e. if  $s_0 \leq s_1 \leq s_2 \leq \dots \leq s_p$  then  $s_j = -s_{p-j}$ ,  $j = 1, 2, \dots, (p+1)/2$ . Since  $\omega = (\xi/s)^2$ , it is the same for the positive and for the negative  $s$ . Therefore the energies which correspond to the cases analytically solvable, for given  $l$  and  $p$ , are the same for harmonium and for the confined positronium. A plot of spectra

of harmonium ( $\xi = 1$ ) and of the confined positronium ( $\xi = -1$ ), versus  $\log \omega$ , is presented in Fig. 1. The spectra are scaled by the excitation energy to the 10th excited state of the confined positronium, i.e. the quantities plotted are equal to

$$\Delta_N = \frac{E_N - E_1^{\text{cps}}}{E_{10}^{\text{cps}} - E_1^{\text{cps}}}, \quad (33)$$

where the superscript cps refers to the confined positronium and  $E_N$  is the energy of the Nth energy level of either harmonium or confined positronium. The energies have been obtained by a numerical integration of Eq. (9). The regularities in the distribution of the crossing points which may be seen in Fig. 1 are also illustrated in Tables 2 and 3. The values of  $\omega$

Table 1. Values of  $s = \xi$  (under assumption of  $\omega = 1$ ) and of  $1/s^2 = \omega$  (under assumption of  $\xi = 1$ ) for the confined positronium (cps) and for harmonium (hrm) in the case of  $l = 0$ ;  $N$  denotes the consecutive number of the energy level of the pertinent system ( $N = 1$  stands for the ground state,  $N = 2$  - for the first excited state, etc.). In the last column the corresponding energies are displayed

$p$	System	$N$	$\zeta (\omega = 1)$	$\omega ( \zeta  = 1)$	$E_N^r$
1	cps	2	-1.4142135623731	0.5000000000000	1.2500000000000
	hrm	1	1.4142135623731	0.5000000000000	1.2500000000000
2	cps	3	-3.1622776601684	0.1000000000000	0.3500000000000
	hrm	2	0.0000000000000	arbitrary	
3	hrm	1	3.1622776601684	0.1000000000000	0.3500000000000
	cps	4	-5.2315692556682	0.0365372655993	0.1644176951967
	cps	3	-1.6219380762369	0.3801294010674	1.7105823048033
	hrm	2	1.6219380762369	0.3801294010674	1.7105823048033
4	hrm	1	5.2315692556682	0.0365372655993	0.1644176951967
	cps	5	-7.5927270006113	0.0173462032222	0.0954041177220
	cps	4	-3.5143273459067	0.0809684035194	0.4453262193567
	hrm	3	0.0000000000000	arbitrary	
5	hrm	2	3.5143273459067	0.0809684035194	0.4453262193567
	hrm	1	7.5927270006113	0.0173462032222	0.0954041177220
	cps	6	-10.2176937143200	0.0095784280156	0.0622597821011
	cps	5	-5.6926771788333	0.0308579369294	0.2005765900409
	cps	4	-1.7866621667052	0.3132673387588	2.0362377019320
	hrm	3	1.7866621667052	0.3132673387588	2.0362377019320
6	hrm	2	5.6926771788333	0.0308579369294	0.2005765900409
	hrm	1	10.2176937143200	0.0095784280156	0.0622597821011
	cps	7	-13.0836896931990	0.0058417037553	0.0438127781646
	cps	6	-8.1436471236368	0.0150786377025	0.1130897827687
	cps	5	-3.8076338502751	0.0689746716656	0.5173100374919
	hrm	4	0.0000000000000	arbitrary	
7	hrm	3	3.8076338502751	0.0689746716656	0.5173100374919
	hrm	2	8.1436471236368	0.0150786377025	0.1130897827687
	hrm	1	13.0836896931990	0.0058417037553	0.0438127781646
	cps	8	-16.1725422920744	0.0038233443007	0.0324984265556
	cps	7	-10.8466887415847	0.0084997400645	0.0722477905482
	cps	6	-6.0900495359691	0.0269623877262	0.2291802956728
8	cps	5	-1.9260103332246	0.2695769617711	2.2914041750541
	hrm	4	1.9260103332246	0.2695769617711	2.2914041750541
	hrm	3	6.0900495359691	0.0269623877262	0.2291802956728
	hrm	2	10.8466887415847	0.0084997400645	0.0722477905482
	hrm	1	16.1725422920744	0.0038233443007	0.0324984265556

Table 1 continued

$p$	System	$N$	$\zeta (\omega = 1)$	$\omega ( \zeta  = 1)$	$E_{nl}^*$
8	cps	9	-19.4695097841489	0.0026380921801	0.0250618757111
	cps	8	-13.7826923466140	0.0052641938792	0.0500098418523
	cps	7	-8.6296680009879	0.0134280151982	0.1275661443829
	cps	6	-4.0625620778327	0.0605898642514	0.5756037103886
		5	0.0000000000000	arbitrary	
	hrm	4	4.0625620778327	0.0605898642514	0.5756037103886
	hrm	3	8.6296680009879	0.0134280151982	0.1275661443829
	hrm	2	13.7826923466140	0.0052641938792	0.0500098418523
9	hrm	1	19.4695097841489	0.0026380921801	0.0250618757111
	cps	10	-22.9623769219761	0.0018965588222	0.0199138676329
	cps	9	-16.9355445243392	0.0034865963411	0.0366092615815
	cps	8	-11.4111143045907	0.0076796935197	0.0806367819566
	cps	7	-6.4426385912978	0.0240919781510	0.2529657705854
	cps	6	-2.0482812469742	0.2383531096740	2.5027076515768
	hrm	5	2.0482812469742	0.2383531096740	2.5027076515768
	hrm	4	6.4426385912978	0.0240919781510	0.2529657705854
	hrm	3	11.4111143045907	0.0076796935197	0.0806367819566
	hrm	2	16.9355445243392	0.0034865963411	0.0366092615815
10	hrm	1	22.9623769219762	0.0018965588222	0.0199138676329
	cps	11	-26.6408261152537	0.0014089793372	0.0162032623777
	cps	10	-20.2918044638954	0.0024286149414	0.0279290718265
	cps	9	-14.4181055196040	0.0048104266936	0.0553199069762
	cps	8	-9.0676588535791	0.0121621303801	0.1398644993717
	cps	7	-4.2900876024306	0.0543334996526	0.6248352460052
		6	0.0000000000000	arbitrary	
	hrm	5	4.2900876024306	0.0543334996526	0.6248352460052
	hrm	4	9.0676588535791	0.0121621303801	0.1398644993717
	hrm	3	14.4181055196040	0.0048104266936	0.0553199069762
	hrm	2	20.2918044638954	0.0024286149414	0.0279290718265
	11	hrm	1	26.6408261152537	0.0014089793372
cps		12	-30.4960000236087	0.0010752618599	0.0134407732483
cps		11	-23.8401464471275	0.0017594711990	0.0219933899875
cps		10	-17.6362253879963	0.0032150573126	0.0401882164078
cps		9	-11.9258755355747	0.0070310381592	0.0878879769895
cps		8	-6.7617284108466	0.0218718014053	0.2733975175658
cps		7	-2.1581185350441	0.2147085853920	2.6838573173995
hrm		6	2.1581185350441	0.2147085853920	2.6838573173995
hrm		5	6.7617284108466	0.0218718014053	0.2733975175658
hrm		4	11.9258755355747	0.0070310381592	0.0878879769895
hrm		3	17.6362253879963	0.0032150573126	0.0401882164078
hrm		2	23.8401464471275	0.0017594711990	0.0219933899875
hrm		1	30.4960000236087	0.0010752618599	0.0134407732483

corresponding to the crossing points in Fig. 1 are ordered according to the values of  $N_h$ ,  $N_p$  and  $p$  for  $1 \leq p \leq 20$  in Table 2. The corresponding values of the indices  $(N_h, N_p, p)$  are displayed in Table 3. As one can see, if  $\zeta = \pm 1$  then

$$0 < \omega(N_h, N_p, p) < \omega(N_h + 1, N_p, p + 1) \quad 0.5,$$

$$0 < \omega(N_h, N_p + 1, p + 1) < \omega(N_h, N_p, p) \quad 0.5,$$

$$0 < \omega(N_h + 1, N_p + 1, p + 2) < \omega(N_h, N_p, p) \quad 0.5.$$



Each of the inequalities links the values of  $co$  distributed along a specific line in Fig. 1. In particular, the last inequality corresponds to the series connected by the dotted lines in Fig. 1 and displayed in the same row of Table 2.

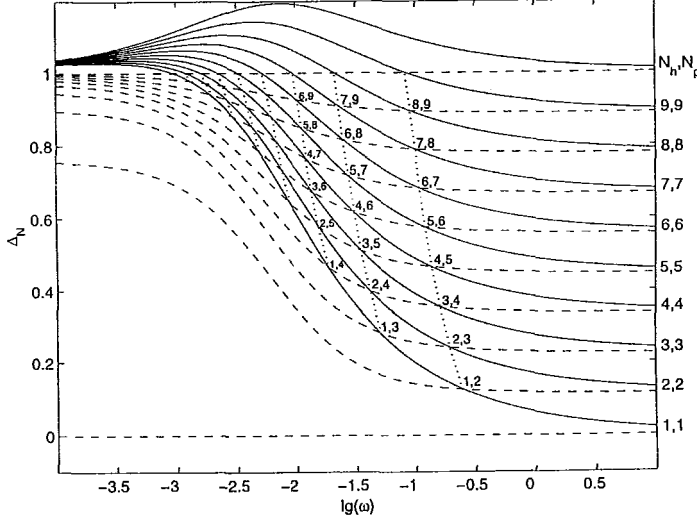


Fig. 1. Energy levels of harmonium (solid lines) and of the confined positronium (broken lines) for  $l = 0$  scaled by the excitation energy to the 10th excited state of the confined positronium, versus  $\log \omega$ . The points where the energy levels cross correspond to the analytical solutions of Eq. (9) with  $\zeta = \pm 1$ . The crossing points are labelled by  $(N_h, N_p)$ , where  $N_h, N_p = 1, 2, 3, \dots$  correspond to the consecutive number of the energy level in the spectrum of, respectively, harmonium and positronium. The points corresponding to  $N_p - N_h = n_0$  (in Table 2 they are collected in the same row) are connected by dotted lines (the rightmost line corresponds to  $n_0 = 1$ , and the consecutive ones to  $n_0 = 2, 3, \dots$ ).

If  $p$  is even, i.e. the dimension of  $T$  is odd, then  $s_{p/2} = 0$ . This case corresponds to the unperturbed spherical harmonic oscillator. The well known analytical solutions

$$P_{nl}^{(0p)} \sim (-p/2, l + 3/2, \omega\rho^2), \tag{34}$$

where  $F$  is the confluent hypergeometric function, contain only even powers of  $\rho$ . Therefore the case of  $s_{p/2} = 0$  appears for every second (even) value of  $p$ . Let us note that the asymptotic behaviour of  $\phi_{nl}^s$  is  $y$ -independent [cf. Eq. (11)]. Therefore the functions  $\phi_{nl}^s$  may always be represented as linear combinations of  $\phi_{kl}^0$  and,  $\phi_{k,l+1}^0, k = 0, 1, \dots, p/2$ . In particular for  $p = 1$

$$\phi_{0l}^s \sim \phi_{0l}^0 + \sqrt{1 + 2\omega_1} \phi_{0,l+1}^0,$$

where  $\omega_1 = 1/[4(l + 1)]$ , and for  $p = 2$

$$\phi_{0l}^s \sim \phi_{0l}^0 + \sqrt{1 + 4\omega_2} \phi_{0,l+1}^0 + \sqrt{4\omega_2(1 + 4\omega_2)} \phi_{1l}^0,$$

where  $\omega_2 = 1/[4(4l + 5)]$ .

Table 2. Values of  $1000/s^2 = 1000 \times \omega$  in the case of  $l = 0$  and  $|\xi| = 1$  ordered according to  $N_h, N_p,$  and  $p$ . The corresponding values of  $(N_h, N_p, p)$  are given in Table 3

0.1923									
0.2232									
0.2611	0.2507								
0.3082	0.2952								
0.3672	0.3509	0.3363							
0.4424	0.4216	0.4031							
0.5396	0.5127	0.4889	0.4676						
0.6675	0.6320	0.6008	0.5732						
0.8392	0.7914	0.7498	0.7133	0.6809					
1.0753	1.0094	0.9527	0.9033	0.8598					
1.4090	1.3155	1.2361	1.1676	1.1078	1.0550				
1.8966	1.7595	1.6446	1.5468	1.4621	1.3881				
2.6381	2.4286	2.2561	2.1111	1.9872	1.8798	1.7856			
3.8233	3.4866	3.2151	2.9905	2.8011	2.6388	2.4977			
5.8417	5.2642	4.8104	4.4426	4.1372	3.8787	3.6565	3.4630		
9.5784	8.4997	7.6797	7.0310	6.5025	6.0620	5.6879	5.3656		
17.3462	15.0786	13.4280	12.1621	11.1544	10.3293	9.6389	9.0509	8.5430	
36.5373	30.8579	26.9624	24.0920	21.8718	20.0933	18.6303	17.4015	16.3520	
100.0000	80.9684	68.9747	60.5899	54.3335	49.4514	45.5149	42.2604	39.5163	37.1652
500.0000	380.1294	313.2673	269.5770	238.3531	214.7086	196.0616	180.9066	168.3003	157.6185

Table 3. The indices  $(N_h, N_p, p)$  corresponding to the values of  $\omega$  displayed in Table 2

(1,21,20)									
(1,20,19)									
(1,19,18)	(2,20,20)								
(1,18,17)	(2,19,19)								
(1,17,16)	(2,18,18)	(3,19,20)							
(1,16,15)	(2,17,17)	(3,18,19)							
(1,15,14)	(2,16,16)	(3,17,18)	(4,18,20)						
(1,14,13)	(2,15,15)	(3,16,17)	(4,17,19)						
(1,13,12)	(2,14,14)	(3,15,16)	(4,16,18)	(5,17,20)					
(1,12,11)	(2,13,13)	(3,14,15)	(4,15,17)	(5,16,19)					
(1,11,10)	(2,12,12)	(3,13,14)	(4,14,16)	(5,15,18)	(6,16,20)				
(1,10,9)	(2,11,11)	(3,12,13)	(4,13,15)	(5,14,17)	(6,15,19)				
(1,9,8)	(2,10,10)	(3,11,12)	(4,12,14)	(5,13,16)	(6,14,18)	(7,15,20)			
(1,8,7)	(2,9,9)	(3,10,11)	(4,11,13)	(5,12,15)	(6,13,17)	(7,14,19)			
(1,7,6)	(2,8,8)	(3,9,10)	(4,10,12)	(5,11,14)	(6,12,16)	(7,13,18)	(8,14,20)		
(1,6,5)	(2,7,7)	(3,8,9)	(4,9,11)	(5,10,13)	(6,11,15)	(7,12,17)	(8,13,19)		
(1,5,4)	(2,6,6)	(3,7,8)	(4,8,10)	(5,9,12)	(6,10,14)	(7,11,16)	(8,12,18)	(9,13,20)	
(1,4,3)	(2,5,5)	(3,6,7)	(4,7,9)	(5,8,11)	(6,9,13)	(7,10,15)	(8,11,17)	(9,12,19)	
(1,3,2)	(2,4,4)	(3,5,6)	(4,6,8)	(5,7,10)	(6,8,12)	(7,9,14)	(8,10,16)	(9,11,18)	(10,12,20)
(1,2,1)	(2,3,3)	(3,4,5)	(4,5,7)	(5,6,9)	(6,7,11)	(7,8,13)	(8,9,15)	(9,10,17)	(10,11,19)

Equation (9) for harmonium reads

$$\left[ H_0 + \frac{|s|}{\rho} \right] \phi_{nl}^{|s|}(\rho) = \varepsilon_{nl}^+ \phi_{nl}^{|s|}(\rho), \tag{35}$$

where

$$H_0 = -\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2} + \frac{1}{4}\rho^2 \quad (36)$$

and for the confined positronium it is

$$\left[ H_0 - \frac{|s|}{\rho} \right] \phi_{nl}^{-|s|}(\rho) = \varepsilon_{nl}^- \phi_{nl}^{-|s|}(\rho), \quad (37)$$

with  $\rho \in \langle 0, \infty \rangle$ . If  $s$  is chosen to be an eigenvalue of  $T$ , i.e.  $\phi$  is an analytical solution (11) of Eq. (9), then  $\varepsilon_{nl}^+ = \varepsilon_{nl}^-$ . Consequently, by comparing Eqs. (35) and (37), we see that

$$\phi_{nl}^{|s|}(\rho) = \phi_{nl}^{-|s|}(-\rho). \quad (38)$$

Then, by taking  $\rho \in \langle -\infty, \infty \rangle$ , Eq. (35) describes both harmonium and confined positronium with  $\phi_{nl}^{|s|}(\rho)$  corresponding to harmonium if  $\rho \geq 0$  and to the confined positronium if  $\rho \leq 0$ .

Equations (35) and (37), with  $\rho \geq 0$ , may also be expressed as a single two-component eigenvalue equation:

$$\begin{pmatrix} H_0 & s/r \\ s/r & H_0 \end{pmatrix} \begin{pmatrix} \psi_{nl}^+ \\ \psi_{nl}^- \end{pmatrix} = \varepsilon_{nl} \begin{pmatrix} \psi_{nl}^+ \\ \psi_{nl}^- \end{pmatrix}, \quad (39)$$

where  $\varepsilon_{nl} = \varepsilon_{nl}^+ = \varepsilon_{nl}^-$ ,

$$\begin{aligned} \psi_{nl}^+ &\sim \phi_{nl}^{|s|} + \phi_{nl}^{-|s|} = \rho^{l+1} e^{-\rho^2/4} P_{nl}^+(\rho) \\ \psi_{nl}^- &\sim \phi_{nl}^{|s|} - \phi_{nl}^{-|s|} = \rho^{l+1} e^{-\rho^2/4} P_{nl}^-(\rho) \end{aligned}$$

with

$$\begin{aligned} P_{nl}^+(\rho) &= P_{nl}(\rho) + P_{nl}(-\rho) \\ P_{nl}^-(\rho) &= P_{nl}(\rho) - P_{nl}(-\rho). \end{aligned}$$

Let us note that  $P_{nl}^+(\rho)$  and  $P_{nl}^-(\rho)$  are, respectively, an even and an odd polynomial in  $\rho$ . Hence, if  $\psi_{nl}^+$  may be expressed as a linear combination of  $\phi_{kl}^0$ ,  $k = 0, 1, \dots, p/2$  then  $\psi_{nl}^-$  is expressible as a linear combination of  $\phi_{n,l+1}^0$ .

## 6. CONCLUDING REMARKS

The analytical solutions of the eigenvalue equation of the Schrödinger Hamiltonian describing two particles confined in a parabolic potential well and interacting by a Coulomb force exhibit many remarkable properties. By these properties they differ in an essential way from the other solutions of this equation. In particular, only in the case of the analytical solutions the transformation  $\rho \leftrightarrow -\rho$  corresponds to the transition (harmonium)  $\leftrightarrow$  (confined positronium). Also only for the analytical solutions the energy of harmonium and of confined

positronium is, for a given value of  $|s|$ , the same. These properties suggest that the existence of the analytical solutions is associated with a hidden symmetry of this system.

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