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IMPLICIT INTERVAL MULTISTEP METHODS FOR SOLVING THE INITIAL VALUE PROBLEM

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Abstract. Implicit interval methods of Adams-Moulton type for solving the initial value problem are proposed. It is proved that the exact solution of the problem belongs to interval-solutions obtained by the considered methods. Furthermore, the widths of interval-solutions are estimated.

1. INTRODUCTION

In this paper we direct our attention to the interval multistep methods for solving the initial value problem. Interval numerical methods are very interesting due to interval-solutions obtained by such methods which contain their errors.

Using a computer implementation of the interval methods in floating-point interval arithmetic together with the representation of initial data in the form of minimal machine intervals, i.e. by intervals which ends are equal or neighbouring machine numbers, let us achieve interval-solutions which contain all possible numerical errors.

Explicit interval multistep methods of Adams-Bashforth type have been considered by Šokin [6, 16]. As we know from the analysis of conventional methods, higher orders of accuracy can be achieved by applying implicit methods rather then explicit one. For this reason implicit interval multistep methods of Adams-Moulton type are the subject of our present research.

This paper consists of six sections. In Sec. 2 we define the initial value problem and present the conventional implicit Adams-Moulton methods whose interval equivalents are introduced in Sec. 3. We also prove that the exact solution of the initial value problem belongs to interval-solutions obtained by the interval Adams-Moulton method (Sec. 4). At the end of our paper we estimate the widths of interval-solutions obtained by the interval methods considered (Sec. 5) and draw some remarks (Sec. 6).

2. THE INITIAL VALUE PROBLEM AND CONVENTIONAL ADAMS-MOULTON METHODS

As we know the initial value problem is concerned with finding the solution $y = y(t)$ to a problem of the form

$$y'(t) = f(t, y(t)) \tag{2.1}$$

subject to an initial condition

$$y(0) = y_0,$$

where $t \in [0, \xi]$, $y \in \mathbf{R}^N$ and $f : [0, \xi] \times \mathbf{R}^N \to \mathbf{R}^N$. We will assume that the solution of (2.1) exists and is unique. The theory of ordinary differential equations states that these conditions are fulfilled if the function f is determined and continuous in the set $\{(t, y): 0 \le t \le \xi,\ y \in \mathbf{R}^N\}$ and there exists a constant $L > 0$ such that for each $t \in [0, \xi]$ and all $y_1, y_2 \in \mathbf{R}^N$ we have

$$\| f(t, y_1) - f(t, y_2) \| \le L \| y_1 - y_2 \|.$$

Let us choose a positive integer m and select the mesh points $t_0, t_1, \ldots, t_m$ where t_n - nh for each $n = 0, 1, \ldots, m$ and $h = \xi / m$. Assume that an integer $k = 1, 2, \ldots$ will state how many approximations y_{n-k}, $y_{n-k+1}, \ldots,$ y_{n-1} of the exact solution at the previous k mesh points have to be known to determine the approximation y_n at the point t_n.

As shown in [7] the exact solution of (1) considered on the interval $[t_{n-1}, t_n]$ has the form

$$y(t_n) = y(t_{n-1}) + h \sum_{j=0}^{k} \bar{\gamma}_j \nabla^j f(t_n, y(t_n)) + h^{k+2} \bar{\gamma}_{k+1} \psi(\eta, y(\eta)), \tag{2.2}$$

where

$$\nabla^j f(t_n, y(t_n)) = \sum_{m=0}^{j} (-1)^m \binom{j}{m} f(t_{n-m}, y(t_{n-m})),$$

$$\bar{\gamma}_0 = 1, \quad \bar{\gamma}_j = \frac{1}{j!} \int_{-1}^{0} s(s+1)\ldots(s+j-1)\,ds \quad \text{for } j = 1, 2, \ldots, k+1, \tag{2.3}$$

and $\psi(\eta, y(\eta)) \equiv f^{(k+1)}(\eta, y(\eta)) \equiv y^{(k+2)}(\eta)$, $\eta \in [t_{n-k}, t_n]$.

After replacing the unknown values $y(t_{n-k})$, $y(t_{n-k+1})$, ..., $y(t_n\text{-}1)$ approximations y_{n-k}, y_{n-k+1}, ..., y_{n-1} obtained by applying another method (for example by a Runge-Kutta method) and neglecting the error term $h^{k+2-}\gamma_{k+1}\psi(\eta, y(\eta))$ we are given the following formula known as the k-step implicit Adams-Moulton method

$$y_n = y_{n-1} + h \sum_{j=0}^{k} \bar{\gamma}_j \nabla^j f_n, \tag{2.4}$$

where $f_n = f(t_n, y_n)$. Let us note that in general to apply (2.4) we need to solve the system of nonlinear equations with respect to the unknowns y_n.

3. IMPLICIT INTERVAL MULTISTEP METHODS OF ADAMS-MOULTON TYPE

Let us denote:

Δ_t and Δ_y - sets in which the function *f(t, y)* is defined, i.e.

$$\Delta_t = \{ t \in \mathbf{R}:\ 0 \le t \le \xi \},$$
$$\Delta_y = \{ y = (y_1, y_2, \ldots, y_N)^T \in \mathbf{R}^N:\ \underline{b}_i \le y_i \le \overline{b}_i, \quad i = 1, 2, \ldots, N \},$$

F(T ,Y) - an interval extension of *f(t, y)* (for a definition of interval extension see e.g. [4], [13] or [16]),

$\Psi(T, Y)$ - an interval extension of $\psi(t, y)$.

Let us assume that

- the function *F(T,Y)* is defined and continues for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- the function *F(T,Y)* is monotonic with respect to inclusion, i.e.

$$T_1 \subset T_2 \wedge Y_1 \subset Y_2 \Rightarrow F\left(T_1, Y_1\right) \subset F\left(T_2, Y_2\right),$$

- for each $T \subset \Delta_t$ and for each $Y \subset \Delta_y$ there exists a constant $L > 0$ such that

$$d\left(F\left(T, Y\right)\right) \le L\left(d\left(T\right) + d\left(Y\right)\right),$$

where *d(A)* denotes the width of *A* (if $A = (A_1, A_2, \ldots, A_N)^T$, then the number *d(A)* is defined by $d(A) = \max_{i=1,2,\ldots,N} d\left(A_i\right)$,

- the function $\Psi(T, Y)$ is defined for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- the function $\Psi(T, Y)$ is monotonic with respect to inclusion.

Now, after making the above assumptions, we are able to construct the implicit multistep interval method of Adams-Moulton type. First, let us assume that $y(0) \in Y_0$ and the intervals Y_i such as $y\left(t_i\right) \in Y_i$ for $i = 1, 2, \ldots, k - 1$ are known. We can obtain such Y_i- by applying an interval one-step method, for example an interval method of Runge-Kutta type (see [3], [10] or [11]). Then, the implicit interval method of Adams-Moulton type we define as follows

$$\begin{aligned} Y_n = Y_{n-1} &+ h\left(\overline{\gamma}_0 F_n + \overline{\gamma}_1 \nabla F_n + \overline{\gamma}_2 \nabla^2 F_n + \ldots + \overline{\gamma}_k \nabla^k F_n\right) \\ &+ h^{k+2} \overline{\gamma}_{k+1} \Psi\left(T_n + [-kh, 0],\ Y_n + [-kh, 0] F\left(\Delta_t, \Delta_y\right)\right) \end{aligned} \tag{3.1}$$

for $n - k, k + 1, \ldots, m,$ where

$$F_n = F\left(T_n, Y_n\right),$$

$$\overline{\gamma}_0 = 1, \quad \overline{\gamma}_j = \frac{1}{j!} \int_{-1}^{0} s(s+1)\ldots(s+j-1)\, ds, \quad j = 1, 2, \ldots, k+1,$$

$$h = \frac{\xi}{m}, \quad t_i = ih \in T_i, \quad i = 0, 1, \ldots, m.$$

In particular for a given k we get the following methods:

- $k = 1 \Rightarrow$

$$Y_n = Y_{n-1} + \frac{h}{2}\Big(F(T_n, Y_n) + F(T_{n-1}, Y_{n-1})\Big)$$
$$-\frac{h^3}{12}\Psi\Big(T_n + [-h, 0], Y_n + [-h, 0]F(\Delta_t, \Delta_y)\Big),$$

- $k = 2 \Rightarrow$

$$Y_n = Y_{n-1} + \frac{h}{12}\Big(5F(T_n, Y_n) + 8F(T_{n-1}, Y_{n-1}) - F(T_{n-2}, Y_{n-2})\Big)$$
$$-\frac{h^4}{24}\Psi\Big(T_n + [-2h, 0],\ Y_n + [-2h, 0]F(\Delta_t, \Delta_y)\Big),$$

- $k = 3 \Rightarrow$

$$Y_n = Y_{n-1} + \frac{h}{24}\Big(9F(T_n, Y_n) + 19F(T_{n-1}, Y_{n-1}) - 5F(T_{n-2}, Y_{n-2}) + F(T_{n-3}, Y_{n-3})\Big)$$
$$-\frac{19h^5}{720}\Psi\Big(T_n + [-3h, 0],\ Y_n + [-3h, 0]F(\Delta_t, \Delta_y)\Big).$$

The formula (3.1) can be written in the equivalent form

$$Y_n = Y_{n-1} + h\sum_{j=0}^{k}\bar{\gamma}_j \nabla^j F(T_n, Y_n)$$
$$+ h^{k+2}\bar{\gamma}_{k+1}\Psi\Big(T_n + [-kh, 0], Y_n + [-kh, 0]F(\Delta_t, \Delta_y)\Big) \quad (3.2)$$

for $n = k, k+1, \ldots, m$, where

$$\nabla^j F(T_n, Y_n) = \sum_{m=0}^{j}(-1)^m\binom{j}{m}F(T_{n-m}, Y_{n-m}), \quad j = 0, 1, \ldots, k. \quad (3.3)$$

Let us note that (3.2) is a nonlinear interval equation with respect to Y_n ($n = k, k+1, \ldots, m$). It follows that in each step of the method we have to solve an interval equation of the form

$$Y = G(T, Y),$$

where

$$T \in I(\Delta_t) \subset I(\mathbf{R}), \quad Y = (Y_1, Y_2, \ldots, Y_N)^T \in I(\Delta_y) \subset I(\mathbf{R}^N),$$
$$G: I(\Delta_t) \times I(\Delta_y) \to I(\mathbf{R}^N).$$

If we assume that the function G is a contraction mapping, then the well-known fixed-point theorem (see e.g. [9] or [14]) implies that the iteration process

$$Y^{(l+1)} = G\left(T, Y^{(l)}\right), \quad l = 0, 1, \ldots, \tag{3.4}$$

is convergent to Y^*, i.e. $\lim\limits_{l \to \infty} Y^{(l)} = Y^*$, for an arbitrary choice of $Y^{(0)} \in I\left(\Delta_y\right)$. Let us recall that a function G is called a contraction mapping if

$$\rho\left(G(T, Y),\, G\left(T, \overline{Y}\right)\right) \le \alpha \rho\left(Y, \overline{Y}\right),$$

where p is a metric, and $\alpha < 1$ denotes a constant.

Let us note that using the fact that

$$\nabla^j F\left(T_n, Y_n\right) = \sum_{m=0}^{j} (-1)^m \binom{j}{m} F\left(T_{n-m}, Y_{n-m}\right),$$

the equation (3.1) can be written in the form

$$\begin{aligned} Y_n = Y_{n-1} &+ h\overline{\beta}_{k0} F\left(T_n, Y_n\right) + h \sum_{j=1}^{k} \overline{\beta}_{kj} F\left(T_{n-j}, Y_{n-j}\right) \\ &+ h^{k+2} \overline{\gamma}_{k+1} \Psi\left(T_n + [-kh, 0],\ Y_n + [-kh, 0] F\left(\Delta_t, \Delta_y\right)\right), \end{aligned} \tag{3.5}$$

where

$$\overline{\beta}_{kj} = (-1)^j \sum_{m=j}^{k} \binom{m}{j} \overline{\gamma}_m, \quad j = 0, 1, \ldots, k. \tag{3.6}$$

For the equation (3.5) the iteration process (3.4) is of the form

$$\begin{aligned} Y_n^{(l+1)} = Y_{n-1} &+ h\overline{\beta}_{k0} F\left(T_n, Y_n^{(l)}\right) + h \sum_{j=1}^{k} \overline{\beta}_{kj} F\left(T_{n-j}, Y_{n-j}\right) \\ &+ h^{k+2} \overline{\gamma}_{k+1} \Psi\left(T_n + [-kh, 0],\ Y_n^{(l)} + [-kh, 0] F\left(\Delta_t, \Delta_y\right)\right), \quad l = 0, 1, \ldots, \end{aligned}$$

and we usually choose $Y_n^{(0)} = Y_{n-1}$.

4. THE EXACT SOLUTION VS. INTERVAL SOLUTIONS

For the method (3.2) we can prove that the exact solution of the initial value problem (2.1) belongs to the intervals obtained by this method. Before that it is convenient to present the lemma.

Lemma 1. If $(t_i, y(t_i)) \in (T_i, Y_i)$ for $i = n-k, n-k+1, \ldots, n-1$, where $Y_i = Y(t_i)$, than for any $j = 0,1, \ldots, k-1$, k we have

$$\nabla^j f(t_n, y(t_n)) \in \nabla^j F(T_n, Y_n). \tag{4.1}$$

Proof. Since $F(T, Y)$ is an interval extension of $f(t, y)$, then $f(t, y) \in F(T, Y)$ for each $t \in \Delta_t$ and for each $y \in \Delta_y$. This fact implies that $(t_n, y(t_n)) \in (T_n, Y_n)$, where $Y_n \subset \Delta_y$. Moreover, $(t_i, y(t_i)) \in (T_i, Y_i)$ for $i = n-k$, $n-k+1$, ..., $n-1$, and hence we get the inclusion as follows

$$f(t_{n-m}, y(t_{n-m})) \in F(T_{n-m}, Y_{n-m}), \quad m = 0, 1, \ldots, j.$$

This implies that

$$\sum_{m=0}^{j} (-1)^m \binom{j}{m} f(t_{n-m}, y(t_{n-m})) \in \sum_{m=0}^{j} (-1)^m \binom{j}{m} F(T_{n-m}, Y_{n-m}). \tag{4.2}$$

But

$$\sum_{m=0}^{j} (-1)^m \binom{j}{m} F(T_{n-m}, Y_{n-m}) = \nabla^j F(T_n, Y_n). \tag{4.3}$$

From (2.3), (4.2) and (4.3) the inclusion (4.1) follows immediately.

Theorem 1. If $y(0) \in Y_0$ and $y(t_i) \in Y_i$ for $i = 1, 2, \ldots, k-1$, then for the exact solution $y(t)$ of the initial value problem (2.1) we have

$$y(t_n) \in Y_n$$

for $n = k, k+1, \ldots, m$, where $Y_n = Y(t_n)$ are obtained from the method (3.2).

Proof. Let us consider the formula (2.2) for $n = k$. We get

$$y(t_k) = y(t_{k-1}) + h \sum_{j=0}^{k} \bar{\gamma}_j \nabla^j f(t_k, y(t_k)) + h^{k+2} \bar{\gamma}_{k+1} \psi(\eta, y(\eta)), \tag{4.4}$$

where $\eta \in [t_0, t_k]$. From the assumption we have $y(t_{k-1}) \in Y_{k-1}$, and from the Lemma 1 it follows that

$$h \sum_{j=0}^{k} \bar{\gamma}_j \nabla^j f(t_k, y(t_k)) \in h \sum_{j=0}^{k} \bar{\gamma}_j \nabla^j F(T_k, Y_k).$$

Applying Taylor's formula we have

$$y(\eta) = y(t_k) + y'(t_k + \vartheta(\eta - t_k))(\eta - t_k), \tag{4.5}$$

where $\vartheta \in [0, 1]$. Because $\eta \in [t_0, t_k]$ and $t_i = ih$ for $i = 0, 1, \ldots, m$, we get

$$\eta - t_k \in [-kh, 0]. \tag{4.6}$$

Moreover, $y'(t) = f(t, y(t))$. Since

$$f\left(t_k + \vartheta(\eta - t_k), y\left(t_k + \vartheta(\eta - t_k)\right)\right) \in F\left(\Delta_t, \Delta_y\right),$$

then

$$y'\left(t_k + \vartheta(\eta - t_k)\right) \in F\left(\Delta_t, \Delta_y\right).$$

In addition, $F(T,Y)$ is an interval extension of $f(t, y)$, and hence $f(t, y) \in F(T, Y)$ for each $t \in \Delta_t$ and $y \in \Delta_y$. For these reasons we can state that $y(t_k) \in Y_k$, where $Y_k \subset \Delta_y$. Taking into account the above considerations, from the formula (4.5) we get

$$y(\eta) \in Y_k + [-kh, 0] F\left(\Delta_t, \Delta_y\right). \tag{4.7}$$

As we assumed, Ψ is an interval extension of ψ. Thus, applying (4.6) and (4.7), we have

$$h^{k+2}\bar{\gamma}_{k+1}\psi(\eta, y(\eta)) \in h^{k+2}\bar{\gamma}_{k+1}\Psi\left(T_k + [-kh, 0], Y_k + [-kh, 0] F\left[\Delta_t, \Delta_y\right]\right).$$

Thus, we have shown that $y(t_k)$ belongs to an interval

$$Y_{k-1} + h\sum_{j=0}^{k}\bar{\gamma}_j \nabla^j F\left(T_k, Y_k\right) + h^{k+2}\bar{\gamma}_{k+1}\Psi\left(T_k + [-kh, 0], Y_k + [-kh, 0] F\left(\Delta_t, \Delta_y\right)\right),$$

but - according to the formula (3.2) - this is the interval Y_k. This conclusion ends the proof for $n = k$. In a similar way we show the thesis of this theorem for $n = k + 1, k + 2, \ldots, m$. •

5. WIDTHS OF INTERVAL SOLUTIONS

Theorem 2. If the intervals Y_n for $n = 0, 1, \ldots, k\text{-}1$ are known, $t_i = ih \in T_i$, $i = 0, 1, \ldots, m$,

$$h = \frac{\xi}{m}, \quad 0 < h \le h_0,$$

where

$$h_0 < \frac{1}{L\beta_k}, \quad \beta_k = \max_{j=0,1,\ldots,k}\left|\bar{\beta}_{kj}\right|, \quad L > 0,$$

and Y_n for $n = k, k+1, \ldots, m$ are obtained from (3.2), then

$$d\left(Y_n\right) \le A \max_{q=0,1,\ldots,k-1} d\left(Y_q\right) + B \max_{j=1,2,\ldots,m} d\left(T_j\right) + Ch^{k+1}, \tag{5.1}$$

where the constants A, B and C are independent of h.

Proof. Substituting (3.3) and (3.6) into (3.2) we have

$$Y_n = Y_{n-1} + h \sum_{j=0}^{k} \overline{\beta}_{kj} F\left(T_{n-j}, Y_{n-j}\right)$$
$$+ h^{k+2} \overline{\gamma}_{k+1} \Psi\left(T_n + [-kh, 0], Y_n + [-kh, 0] F\left(\Delta_t, \Delta_y\right)\right).$$

From the above formula we get

$$d\left(Y_n\right) \le d\left(Y_{n-1}\right) + h \sum_{j=0}^{k} \left|\overline{\beta}_{kj}\right| d\left(F\left(T_{n-j}, Y_{n-j}\right)\right)$$
$$+ h^{k+2} \left|\overline{\gamma}_{k+1}\right| d\left(\Psi\left(T_n + [-kh, 0], Y_n + [-kh, 0] F\left(\Delta_t, \Delta_y\right)\right)\right). \tag{5.2}$$

We have assumed that Ψ is monotonic with respect to inclusion. Moreover, if the step size h is such that satisfies the conditions

$$T_n + [-kh, 0] \subset \Delta_t,$$
$$Y_n + [-kh, 0] F\left(\Delta_t, \Delta_y\right) \subset \Delta_y,$$

than

$$\Psi\left(T_n + [-kh, 0], Y_n + [-kh, 0] F\left(\Delta_t, \Delta_y\right)\right) \subset \Psi\left(\Delta_t, \Delta_y\right). \tag{5.3}$$

From (5.3) we have

$$d\left(\Psi\left(T_n + [-kh, 0], Y_n + [-kh, 0] F\left(\Delta_t, \Delta_y\right)\right)\right) \le d\left(\Psi\left(\Delta_t, \Delta_y\right)\right).$$

We have also assumed that for the function F there exists a constant $L > 0$ such that

$$d\left(F\left(T_{n-j}, Y_{n-j}\right)\right) \le L\left(d\left(T_{n-j}\right) + d\left(Y_{n-j}\right)\right).$$

Therefore, from the inequality (5.2) we get

$$d\left(Y_n\right) \le d\left(Y_{n-1}\right) + hL\beta_k \sum_{j=0}^{k} \left(d\left(T_{n-j}\right) + d\left(Y_{n-j}\right)\right) + h^{k+2} \left|\overline{\gamma}_{k+1}\right| d\left(\Psi\left(\Delta_t, \Delta_y\right)\right), \tag{5.4}$$

where

$$\beta_k = \max_{j=0,1,\ldots,k} \left|\overline{\beta}_{kj}\right|.$$

Denoting

$$\beta = hL\beta_k, \quad \alpha = 1 + \beta, \quad \gamma = h^{k+2} \left|\overline{\gamma}_{k+1}\right|, \tag{5.5}$$

we can write (5.4) in the form

$$d\left(Y_n\right) \le d\left(Y_{n-1}\right) + \beta d\left(Y_n\right) + \beta \sum_{j=1}^{k} d\left(Y_{n-j}\right) + \beta \sum_{j=0}^{k} d\left(T_{n-j}\right) + \gamma\, d\left(\Psi\left(\Delta_t, \Delta_y\right)\right),$$

that is equivalent to

$$\left(1-\beta\right) d\left(Y_n\right) \le \alpha \sum_{j=1}^{k} d\left(Y_{n-j}\right) + \beta \sum_{j=0}^{k} d\left(T_{n-j}\right) + \gamma\, d\left(\Psi\left(\Delta_t, \Delta_y\right)\right). \tag{5.6}$$

Let us assume that

$$1-\beta = 1 - hL\beta_k > 0. \tag{5.7}$$

The condition (5.7) is satisfied if

$$0 < h \le h_0,$$

where

$$h_0 < \frac{1}{L\beta_k}.$$

On the basis of the above assumptions the inequality (5.6) can be written in the form

$$d\left(Y_n\right) \le \nu\alpha \sum_{j=1}^{k} d\left(Y_{n-j}\right) + \nu\beta \sum_{j=0}^{k} d\left(T_{n-j}\right) + \nu\gamma\, d\left(\Psi\left(\Delta_t, \Delta_y\right)\right), \tag{5.8}$$

where

$$\nu = \frac{1}{1 - h_0 L\beta_k}.$$

From (5.8) for $n = k$ we have

$$d\left(Y_k\right) \le \nu\alpha \sum_{j=1}^{k} d\left(Y_{k-j}\right) + \nu\beta \sum_{j=0}^{k} d\left(T_{k-j}\right) + \nu\gamma\, d\left(\Psi\left(\Delta_t, \Delta_y\right)\right), \tag{5.9}$$

and for $n = k+1$ we get

$$d\left(Y_{k+1}\right) \le \nu\alpha\, d\left(Y_k\right) + \nu\alpha \sum_{j=1}^{k-1} d\left(Y_{k-j}\right) + \nu\beta \sum_{j=0}^{k} d\left(T_{k+1-j}\right) + \nu\gamma\, d\left(\Psi\left(\Delta_t, \Delta_y\right)\right).$$

Applying (5.9) to the above inequality we obtain

$$d\left(Y_{k+1}\right) \le \left((\nu\alpha)^2 + \nu\alpha\right) \sum_{j=1}^{k-1} d\left(Y_{k-j}\right) + (\nu\alpha)^2\, d\left(Y_0\right)$$
$$+ \nu\beta\left(\nu\alpha \sum_{j=0}^{k} d\left(T_{k-j}\right) + \sum_{j=0}^{k} d\left(T_{k+1-j}\right)\right) + \nu\gamma\left(\nu\alpha+1\right) d\left(\Psi\left(\Delta_t, \Delta_y\right)\right)$$

$$\leq \left((\nu\alpha)^2 + \nu\alpha\right) \sum_{j=1}^{k} d\left(Y_{k-j}\right)$$
$$+ \nu\beta\left(\nu\alpha \sum_{j=0}^{k} d\left(T_{k-j}\right) + \sum_{j=0}^{k} d\left(T_{k+1-j}\right)\right) + \nu\gamma\left(\nu\alpha + 1\right) d\left(\Psi\left(\Delta_t, \Delta_y\right)\right). \tag{5.10}$$

From (5.8) for $n = k + 2$ we get

$$d\left(Y_{k+2}\right) \leq \nu\alpha\, d\left(Y_{k+1}\right) + \nu\alpha\, d\left(Y_k\right) + \nu\alpha \sum_{j=1}^{k-2} d\left(Y_{k-j}\right)$$
$$+ \nu\beta \sum_{j=0}^{k} d\left(T_{k+2-j}\right) + \nu\gamma\, d\left(\Psi\left(\Delta_t, \Delta_y\right)\right).$$

Insertion of (5.9) and (5.10) into the above inequality yields

$$d\left(Y_{k+2}\right) \leq \left((\nu\alpha)^3 + 2(\nu\alpha)^2 + \nu\alpha\right) \sum_{j=1}^{k} d\left(Y_{k-j}\right)$$
$$+ \nu\beta\left(\left((\nu\alpha)^2 + \nu\alpha\right) \sum_{j=0}^{k} d\left(T_{k-j}\right) + \nu\alpha \sum_{j=0}^{k} d\left(T_{k+1-j}\right) + \sum_{j=0}^{k} d\left(T_{k+2-j}\right)\right) \tag{5.11}$$
$$+ \nu\gamma\left((\nu\alpha)^2 + 2\nu\alpha + 1\right) d\left(\Psi\left(\Delta_t, \Delta_y\right)\right).$$

Now, from (5.8) for $n = k + 3$ we get

$$d\left(Y_{k+3}\right) \leq \nu\alpha\, d\left(Y_{k+2}\right) + \nu\alpha\, d\left(Y_{k+1}\right) + \nu\alpha\, d\left(Y_k\right) + \nu\alpha \sum_{j=1}^{k-3} d\left(Y_{k-j}\right)$$
$$+ \nu\beta \sum_{j=0}^{k} d\left(T_{k+3-j}\right) + \nu\gamma\, d\left(\Psi\left(\Delta_t, \Delta_y\right)\right).$$

Applying (5.9), (5.10) and (5.11) to the above formula we have

$$d\left(Y_{k+3}\right) \leq \left((\nu\alpha)^4 + 3(\nu\alpha)^3 + 3(\nu\alpha)^2 + \nu\alpha\right) \sum_{j=1}^{k} d\left(Y_{k-j}\right)$$
$$+ \nu\beta\left(\left((\nu\alpha)^3 + 2(\nu\alpha)^2 + \nu\alpha\right) \sum_{j=0}^{k} d\left(T_{k-j}\right) + \left((\nu\alpha)^2 + \nu\alpha\right) \sum_{j=0}^{k} d\left(T_{k+1-j}\right)\right.$$

$$+ \nu\alpha \sum_{j=0}^{k} d\left(T_{k+2-j}\right) + \sum_{j=0}^{k} d\left(T_{k+3-j}\right)\Bigg) \qquad (5.12)$$

$$+\nu\gamma\left((\nu\alpha)^3 + 3(\nu\alpha)^2 + 3\nu\alpha + 1\right) d\left(\Psi\left(\Delta_t, \Delta_y\right)\right).$$

Thus, for each $i = 0, 1, \ldots, m - k$ we have

$$d\left(Y_{k+i}\right) \le \left(\sum_{l=0}^{i} \binom{i}{l} (\nu\alpha)^{l+1}\right)\left(\sum_{j=1}^{k} d\left(Y_{k-j}\right)\right)$$

$$+ \nu\beta \sum_{p=0}^{i} \left(\sum_{l=0}^{p-1} \binom{p-1}{l} (\nu\alpha)^{l+1}\right)\left(\sum_{j=0}^{k} d\left(T_{k+i-p-j}\right)\right)$$

$$+ \nu\gamma \sum_{l=0}^{i} \binom{i}{l} (\nu\alpha)^{l} d\left(\Psi\left(\Delta_t, \Delta_y\right)\right).$$

Applying the notation (5.5) we obtain

$$d\left(Y_{k+i}\right) \le k \sum_{l=0}^{i} \binom{i}{l} \left(\nu\left(1 + hL\beta_k\right)\right)^{l+1} \max_{q=0,1,\ldots,k-1} d\left(Y_q\right)$$

$$+ \nu hL\beta_k (k+1) \sum_{p=0}^{i} \sum_{l=0}^{p-1} \binom{p-1}{l} \left(\nu\left(1 + hL\beta_k\right)\right)^{l+1} \max_{j=0,1,\ldots,k+i} d\left(T_j\right) \qquad (5.13)$$

$$+ \nu h^{k+2} \left|\bar{\gamma}_{k+1}\right| \sum_{l=0}^{i} \binom{i}{l} \left(\nu\left(1 + hL\beta_k\right)\right)^{l} d\left(\Psi\left(\Delta_t, \Delta_y\right)\right).$$

Let us notice that

$$\binom{i}{l} \le i! \le (m-k)! \quad \text{for } l = 0, 1, \ldots, i,$$

$$\binom{p-1}{l} \le p! \le i! \le (m-k)! \quad \text{for } l = 0, 1, \ldots, p-1,$$

and

$$\left(1 + hL\beta_k\right)^{l+1} \le \exp\left((l+1)hL\beta_k\right) \le \exp\left((i+1)hL\beta_k\right)$$

$$\le \exp\left(mhL\beta_k\right) = \exp\left(\xi L\beta_k\right),$$

$$\begin{aligned}\sum_{l=0}^{p-1} \nu^{l+1}\left(1+hL\beta_k\right)^{l+1} &\le \sum_{l=0}^{i-1} \nu^{l+1}\left(1+hL\beta_k\right)^{l+1} \\ &= \nu\left(1+hL\beta_k\right)\sum_{l=0}^{i-1}\left(\nu\left(1+hL\beta_k\right)\right)^{l} \\ &= \nu\left(1+hL\beta_k\right)\frac{\nu^{i}\left(1+hL\beta_k\right)^{i}-1}{\nu\left(1+hL\beta_k\right)-1} \le \frac{\nu^{i+1}\left(1+hL\beta_k\right)^{i+1}-1}{\nu\left(1+hL\beta_k\right)-1} \\ &\le \frac{\nu^{i+1}\exp\left(\left(i+1\right)hL\beta_k\right)-1}{\nu\left(1+hL\beta_k\right)-1} \le \frac{\nu^{m}\exp\left(mhL\beta_k\right)-1}{\nu\left(1+hL\beta_k\right)-1} \\ &= \frac{\nu^{m}\exp\left(\xi L\beta_k\right)-1}{\nu\left(1+hL\beta_k\right)-1} = \frac{\nu^{m}\exp\left(\xi L\beta_k\right)-1}{\nu hL\beta_k+\nu-1} \\ &\le \frac{\nu^{m}\exp\left(\xi L\beta_k\right)-1}{\nu hL\beta_k}.\end{aligned}$$

On the basis of the above we can make the following estimates

$$\begin{aligned}k\sum_{l=0}^{i}\binom{i}{l}\left(\nu\left(1+hL\beta_k\right)\right)^{l+1} &\le m\left(m-k\right)!\sum_{l=0}^{i}\nu^{l+1}\left(1+hL\beta_k\right)^{l+1} \\ &\le m\left(m-k\right)!\sum_{l=0}^{i}\nu^{l+1}\exp\left(\xi L\beta_k\right) \\ &= m\left(m-k\right)!\,\nu\exp\left(\xi L\beta_k\right)\sum_{l=0}^{i}\nu^{l} \\ &\le m\left(m-k\right)!\,\nu\exp\left(\xi L\beta_k\right)\frac{1-\nu^{m}}{1-\nu},\end{aligned}$$

$$\begin{aligned}\left(k+1\right)\sum_{p=0}^{i}\sum_{l=0}^{p-1}\binom{p-1}{l}\nu^{l+1}\left(1+hL\beta_k\right)^{l+1} &\le \left(m+1\right)\left(m-k\right)!\sum_{p=0}^{i}\sum_{l=0}^{p-1}\nu^{l+1}\left(1+hL\beta_k\right)^{l+1} \\ &\le \left(m+1\right)\left(m-k\right)!\sum_{p=0}^{i}\frac{\nu^{m}\exp\left(\xi L\beta_k\right)-1}{\nu hL\beta_k} \\ &\le \left(m+1\right)\left(m-k+1\right)\left(m-k\right)!\,\frac{\nu^{m}\exp\left(\xi L\beta_k\right)-1}{\nu hL\beta_k},\end{aligned}$$

$$\sum_{l=0}^{i}\binom{i}{l}\left(\nu\left(1+hL\beta_k\right)\right)^l \le (m-k)!\sum_{l=0}^{i}\left(\nu\left(1+hL\beta_k\right)\right)^l$$

$$= (m-k)!\,\frac{\nu^{i+1}\left(1+hL\beta_k\right)^{i+1}-1}{\nu\left(1+hL\beta_k\right)-1}$$

$$\le (m-k)!\,\frac{\nu^{i+1}\exp\left((i+1)hL\beta_k\right)-1}{\nu hL\beta_k+\nu-1}$$

$$\le (m-k)!\,\frac{\nu^{m}\exp\left(mhL\beta_k\right)-1}{\nu hL\beta_k+\nu-1}$$

$$= (m-k)!\,\frac{\nu^{m}\exp\left(\xi L\beta_k\right)-1}{\nu hL\beta_k+\nu-1}$$

$$\le (m-k)!\,\frac{\nu^{m}\exp\left(\xi L\beta_k\right)-1}{\nu hL\beta_k}.$$

Thus, from (5.13) we finally get

$$d\left(Y_{k+i}\right)\le A\max_{q=0,1,\ldots,k-1} d\left(Y_q\right)+B\max_{j=0,1,\ldots,m} d\left(T_j\right)+Ch^{k+1}, \tag{5.14}$$

for each $i = 0,1,\ldots, m - k$, where

$$A = m(m-k)!\,\nu\exp\left(\xi L\beta_k\right)\frac{1-\nu^m}{1-\nu},$$

$$B = (m+1)(m-k+1)(m-k)!\left(\nu^m\exp\left(\xi L\beta_k\right)-1\right),$$

$$C = \frac{\left|\bar{\gamma}_{k+1}\right|}{L\beta_k}(m-k)!\left(\nu^m\exp\left(\xi L\beta_k\right)-1\right)d\left(\Psi(\Delta_t,\Delta_y)\right).$$

Taking into account that $T_0 = [0, 0]$, i.e. $d(T_0) = 0$, the inequality (5.1) follows immediately from (5.14). •

6. REMARKS

The main aim of our paper is a short presentation on the numerical analysis of implicit interval methods. As an example we have given interval methods of Adams-Moulton type. At present, efforts are being made to develop an appropriate computer system which would provide interval--solutions of both explicit and implicit interval methods. Such a system would make it possible to provide calculation in standard floating-point arithmetic and in interval floating-point arithmetic together with interval representation of data in the form of machine intervals.

At the moment one of the main problems is concerned with an iteration process used in the implicit methods. Such a process cannot be too complicated and should be possible to apply to a wide range of interval functions.

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