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On Algebras Associated with Integrable Hamiltonian Systems

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Abstract: The aim of this paper is to give a general setting, based on quantum deformations, for the explicit construction of certain classes of integrable Hamiltonian systems.

Key words: Poisson bialgebras, Casimir functions, Hamiltonian systems, integrability, quantum deformations

I. INTRODUCTION

Integrable Hamiltonian systems play a fundamental role in the study and description of physical systems, due to their many interesting properties, both from the mathematical and physical points of view. Indeed, beyond the obvious interest of finding first integrals, the concept of integrability seems necessary for more thorough understanding of the nonintegrability phenomenon. Integrable Hamiltonian systems always have a hidden algebraic structure that is responsible for their integrability. Therefore, the most interesting problem in the study of dynamical systems is to give such a general algebraic structure which provides a hidden treasure. To date, however, there exists no general method for determining whether or not a given system is integrable. Even in the simplest nontrivial case, i.e. in two-degree of freedom Hamiltonian systems, our knowledge is far from the desired goal. In recent years there has been a renewed interest in completely integrable Hamiltonian systems, especially in conjunction with the study of quantum integrable systems and quantum groups.

This paper presents a procedure in order to construct complete integrable Hamiltonian systems with arbitrary many degrees of freedom from a Poisson bialgebra $(\mathcal{F}(\mathcal{L})_\alpha, P, \Delta, \varepsilon, \{,\}_F)$ on symplectic leaves $\mathcal{L}_\alpha$ of a Poisson manifold $(\mathbb{R}^3, P)$. This construction was put into a geometrical perspective in Refs. [1-3]. We modify the algebraic structure of a Poisson bialgebras by considering the deformed coproduct and deformations of bialgebras. These modifications lead to the quantum groups and provide new classes of completely integrable Hamiltonian systems.

II. POISSON BIALGEBRAS AND INTEGRABLE SYSTEMS

First let us recall some algebraic preliminaries. Detailed exposition can be found for example in Refs. [4-6]. A unital associative algebras over K is a linear space A together with two linear maps $m: A\otimes A \to A$ and $\eta;\ K \to A$ so that: $m(m\otimes\mathbf{1}) = m(\mathbf{1}\otimes m)$, and $m(\mathbf{1}\otimes\eta) = m(\eta\otimes\mathbf{1}) = id$. Let (A_1, m_1, η_1) and (A_2, m_2, η_2) be algebras, then the tensor product $A_1\otimes A_2$ is naturally endowed with the structure of an algebra. The multiplication $m_{a_1\otimes A_2}$ is defined by:

$$(a_1\otimes a_2)(b_1\otimes b_2) = (a_1 b_1)\otimes(a_2 b_2).$$

A coalgebra is a triple (A,Δ,ε) with a linear space A over $K, \Delta: A\to A\otimes A$ a linear map called comultiplication and $\varepsilon: A\to K$ a linear morphism called counit with property

(i) $\Delta(ab) = \Delta(a)\Delta(b), \forall a,b\in A$,
(ii) $(\Delta\otimes id)\circ\Delta = (id\otimes)\circ\Delta$,
(iii) $(id\otimes\varepsilon)\circ\Delta = (\varepsilon\otimes id)\circ\Delta = id$.

We note that $A\otimes A$ is both an algebra and a coalgebra of A. The space of smooth functions $\mathcal{F}(N)$ on a symplectic submanifold N is a Poisson algebra. A tensor product of Poisson algebras $\mathcal{F}(N)\otimes\mathcal{F}(N)$ is again a Poisson algebra. We have to define a Poisson structure on $\mathcal{F}(N)\otimes\mathcal{F}(N)$ such that the axioms of Poisson algebra are satisfied. For our purpose the maps are defined as follows. The multiplication $m_{\mathcal{F}\otimes\mathcal{F}}$

$$\Delta(x_i) = x_i\otimes\mathbf{1} + \mathbf{1}\otimes x_i\,. \tag{1}$$

The Poisson structure on $\mathcal{F}(N)$

$$\{f, g\}_\mathcal{F} = \{x_a, x_b\}\frac{\partial f}{\partial x_a}\frac{\partial g}{\partial x_b}. \tag{2}$$

We define the following Poisson bracket on $\mathcal{F}(N)\otimes\mathcal{F}(N)$:

$$\{f\otimes g, h\otimes j\}_{\mathcal{F}\otimes\mathcal{F}} = \{f, g\}_{\mathcal{F}}\otimes g j + f h\otimes\{g, j\}_{\mathcal{F}}. \quad (3)$$

We will say that the set $(\mathcal{F}(N), m, \Delta, \varepsilon, \{,\}_{\mathcal{F}})$ is a Poisson bialgebra. Assume $\mathcal{C}$ is a Casimir for the Poisson manifold $(\mathbb{R}^3, P)$, then for any $h\in\mathcal{F}(\mathbb{R}^3)$ and $N_\alpha = \mathcal{C}^{-1}(\alpha)$, $\alpha\in\mathbb{R}$

$$\{\mathcal{C}, h\}\big| N_\alpha = \{\tilde{\mathcal{C}}, \tilde{h}\}_{\mathcal{F}} = 0. \quad (4)$$

The above relation defines an integrable Hamiltonian system with two degrees of freedom, given by any function $\tilde{h}\in\mathcal{F}(N)$ and with the second integral generated by the Casimir $\mathcal{C}$.

II.1. Bianchi's algebra

In this subsection we write an explicit formula for the Poisson bialgebras structure arising from the Poisson-Lie structure. We start with the Lie algebra *e*(2) [1]: $[e_1, e_2] = e_3$, $[e_2, e_3] = 0$, $[e_3, e_1] = e_2$. This solvable algebra is of the type VII_0 Bianchi's classification and is isomorphic to the Euklidean algebra of the plane. We consider the dual *e*(2)* to *e*(2) equipped with the linear Poisson-Lie structure

$$P = x_3\partial_1\wedge\partial_2 + x_2\partial_3\wedge\partial_1. \quad (5)$$

Poisson bracket in the space of smooth functions on $e(2)^*$ is defined according to the formula: $\{F, G\} = P(dF, dG)$, $F, G\in\mathcal{F}(e(2^*))$. A Casimir of *P* is

$$C = x_2^2 + x_3^2 \quad (6)$$

and $e(2)^*$ decomposes into a foliation by symplectic submanifolds $N_r = C^{-1}(r), r\in\mathbb{R}$. Since the level sets of *C* are circular cylinders, we choose the usual cylinder coordinates: $x_1 = p$, $x_2 = r\sin q$, $x_3 = r\cos q$, $r = \sqrt{C}$. The transformation of the basis in the tangent space is: $\partial_1 = \partial_p$, $\partial_2 = \sin q\partial_r + r^{-1}\cos q\partial_q$, $\partial_3 = \cos q\partial_r - r^{-1}\sin q\partial_q$. Let $h: N_r\to\mathbb{R}$ be any smooth function with $H\big|N_r = h$. If $f = F\big|N_r$, the Poisson bracket $\{f, h\}$ is defined by restriction $\{F, H\}$ to N_r. The Poisson structure $P_{ab}\big|N_r$ is defined by (5). In terms of coordinates (p, q) on N_r the Poisson bracket of *f* and *h* reads (see e.g. [7])

$$\{f, h\} = \{p, q\}_P\left(\frac{\partial f}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial h}{\partial p}\right). \quad (7)$$

An easy calculation shows that $\{p, q\}_P = 1$, hence *p*, *q* are canonical coordinates on N_r (cf. [1]). From (6) we obtain: $x_1 = p$, $x_2 = \sin q$, $x_3 = \cos q$. Setting:

$$f(p, q)\otimes\mathbf{1} = f(p_1, q_1),$$

$$\mathbf{1}\otimes f(p, q) = f(p_2, q_2),$$

$$f(p, q)\otimes h(p, q) = f(p_1, q_1)h(p_2, q_2)$$

we get

$$\Delta(\mathrm{c}) = 1 + \cos(q_2 - q_1). \quad (8)$$

Thus any Hamiltonian system $(\mathbb{R}^4, \omega, H)$ with Hamiltonian $H(p_1, p_2, q_1, q_2) = \Delta(h(p, \sin q, \cos q))$ is completely integrable if $dH\wedge d(\Delta(\mathrm{c}))\neq 0$ where $\omega^{-1} = \{,\}\times 1 + 1\times\{,\}$.

III. MODIFIED ALGEBRAIC STRUCTURES

III.1. Deformed coproduct

Let us first introduce the deformed coproduct

$$\tilde{\Delta}(x) = x\otimes\mathbf{1} + \mathbf{1}\otimes x,\quad \tilde{\Delta}(y) = y\otimes a_1 + a_2\otimes y,$$
$$\tilde{\Delta}(z) = z\otimes a_1 + a_2\otimes z,\quad \lim_{\epsilon\to 0}\left(a_i(x,\epsilon)\right) = 1. \quad (9)$$

According to our procedure in Sec. II we have

$$\tilde{\Delta}(x) = p_1 + p_2,\quad \tilde{\Delta}(y) + e^{\epsilon p_2}\sin q_1 = e^{-\epsilon p_1}\cos q_2,$$
$$\tilde{\Delta}(z) = e^{\epsilon p_2}\cos q_1 + e^{-\epsilon p_1}\cos q_2. \quad (10)$$

The deformed coproduct of the Casimir

$$\tilde{\Delta}(c) = e^{-2\epsilon p_1} + e^{2\epsilon p_2} + 2e^{\epsilon(p_2 - p_1)}\cos(q_2 - q_1), \quad (11)$$

defines again a family of integrable Hamiltonian systems.

III.2. Deformed algebra structure

Now we introduce the deformed Poisson tensor

$$\tilde{P} = \tilde{z}\partial_{\tilde{x}}\wedge\partial_{\tilde{y}} + \tilde{y}\partial_{\tilde{z}}\wedge\partial_{\tilde{x}} + g(x,\epsilon)\partial_{\tilde{y}}\wedge\partial_{\tilde{z}}, \quad (12)$$

where $\tilde{x} = x$, $\tilde{y} = f(x,\epsilon)$, $\lim_{\epsilon\to 0} g = 0$, $\lim_{\epsilon\to 0} f = 1$. A Poisson coalgebra structure is obtained by means of the coproduct

$$\tilde{\Delta}(\tilde{x}) = \tilde{x}\otimes\mathbf{1} + \mathbf{1}\otimes\tilde{x},\quad \tilde{\Delta}(\tilde{y}) = \tilde{y}\otimes e^{\epsilon\tilde{x}/2} + e^{-\epsilon\tilde{x}/2}\otimes\tilde{y},$$
$$\tilde{\Delta}(\tilde{z}) = \tilde{z}\otimes e^{\epsilon\tilde{x}/2} + e^{-\epsilon\tilde{x}/2}\otimes\tilde{z}. \quad (13)$$

A relation that satisfies all the requirements is

$$\tilde{x} = p,\ \tilde{y} = 2\cosh(\epsilon p/2)\sin q,\ \tilde{z} = 2\cosh(\epsilon p/2)\cos q. \quad (14)$$

Hence

$$\tilde{\Delta}(\tilde{c}) = e^{\epsilon p_2}\cosh^2(\epsilon p_1/2) + e^{-\epsilon p_1/2}\cosh^2(\epsilon p_1/2) +$$
$$+\ 2\cosh(\epsilon p_1/2)\cosh(\epsilon p_2/2)e^{\epsilon(p_2 - p_1)}\cos(q_2 - q_1). \quad (15)$$

IV. CONCLUSIONS

1. The procedure to obtain integrable Hamiltonian systems with two degrees of freedom can be generalized to any number of degrees of freedom by making use of the *k*-th coproduct. Letting $\Delta = \Delta_1$ we find $\Delta_{k+1} = (\Delta \otimes id^k) \circ \Delta_k$, that is, diagonalizing on the first factor after applying Δ_k. Hence, for arbitrary $k \geq 2$, we have

$$\Delta_{k-1}(\mu) = \sum_{i=1}^{k} \mu_i,$$
$$\{\Delta_{k-1}(\mu),\ \Delta_{k-1}(\eta)\} = \sum_{i=1}^{k} \{\mu, \eta\}_i,$$

where μ, η are linear coordinates on $e(2)^*$. The integrals of a Hamiltonian system with *n* degrees of freedom are given by n - 1 coproducts of the Casimir $F_i = \Delta_i(c),\ i = 1, 2, \dots, n-1$, and arbitrary Hamiltonian $H = \Delta_{n\text{-}1}(h(\tilde{x}, \tilde{y}, \tilde{z}))$. An easy computation shows that $H, F_1, \dots, F_{n\text{-}1}$ are in involution, and $dH \wedge dF_i \neq 0$, $dF_i \wedge dF_j \neq 0,\ i \neq j$.

2. The coproduct (9):

$$\tilde{\Delta}(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x,\quad \tilde{\Delta}(y) = y \otimes e^{\epsilon x} + e^{-\epsilon x} \otimes y,$$

$$\tilde{\Delta}(z) = z \otimes e^{\epsilon x} + e^{-\epsilon x} \otimes z$$

defines the quantum deformation $U_\epsilon(e(2))$ (cf. [5, 6]).

3. The deformed structure (12) of $e(2) : [\tilde{e}_1, \tilde{e}_1] = \tilde{e}_3$, $[\tilde{e}_3, \tilde{e}_1] = \tilde{e}_2$, $[\tilde{e}_2, \tilde{e}_1] = -\epsilon \sinh(\epsilon \tilde{e}_1)$ with the deformed coproduct (13):

$$\tilde{\Delta}(\tilde{e}_1) = \tilde{e}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{e}_1,$$

$$\tilde{\Delta}(\tilde{e}_2) = \tilde{e}_2 \otimes e^{\epsilon \tilde{e}_1/2} + e^{-\epsilon \tilde{e}_1/2} \otimes e_2,$$

$$\tilde{\Delta}(\tilde{e}_3) = \tilde{e}_3 \otimes e^{\epsilon \tilde{e}_1/2} + e^{-\epsilon \tilde{e}_1/2} \otimes \tilde{e}_3,$$

may be identified with the nonstandard quantum deformation of algebra *e*(2) (cf. [5, 6]).

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