CMST 14(2) 111-122 (2008) DOI:10.12921/cmst.2008.14.02.111-122

COMPUTATIONAL METHODS IN SCIENCE AND TECHNOLOGY **14(2)**, 111-122 (2008)

Solutions of Two-Dimensional Wave Equation using some Form of the Trefftz Functions

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(Received: 14 May 2008; published online: 19 November 2008)

Abstract: The paper presents a specific technique to generate the Trefftz functions for the two-dimensional wave equation. The obtained functions are used to determine approximate solutions of some tested problems. The accuracy of the method is discussed.

Key words: Trefftz functions, inverse operations, wave equation

I. INTRODUCTION

The wave equation is crucial to many domains of technology. We want to get a solution of the wave equation expressed in terms of some functions which satisfy this equation. We present three methods to obtain the Trefftz functions (wave polynomials) for the wave equation. The first method is connected with a generating function, and leads to recurrent formulas for polynomials and their derivatives, while the second method is based on a Taylor series expansion. Both methods lead to the same form of the Trefftz functions [1, 2]. The third method presented here, apart from the Taylor series expansion, additionally uses the inverse Laplace operator, leading to another form of the Trefftz wave functions which are still polynomials.

II. SOLUTION OF THE WAVE EQUATION BY THE WAVE POLYNOMIALS METHOD

Let us consider a non-homogeneous wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+Q(x,y,t)=\frac{\partial^2 u}{\partial t^2} \tag{1}$$

with the general solution given by

$$u(x,y,t)=L^{-1}(0)+L^{-1}(Q), \qquad L=\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x^2} \tag{2}$$

where $L^{-1}(0)$ is the general solution of a homogeneous equation and $L^{-1}(Q)$ is the particular solution of the no homogeneous one. A general solution is approximated by the linear combination of the Trefftz functions $v_n(x,y,t)$ (wave polynomials)

$$\Theta(x,y,t)=\sum_{n=1}^{N}a_n v_n(x,y,t), \tag{3}$$

satisfying the homogenous wave equation. To find unknown coefficients a_n we minimize the functional describing the adjustment of the approximation (in the mean square sense) to the boundary and initial conditions. The formula for the particular solution $L^{-1}(Q)$ is given in [3]. Here we present the method of forming an approximate solution based on the Trefftz functions for the non-homogenous wave equation which seems to be interesting in generating different forms of these functions.

III. GENERATING THE TREFFTZ FUNCTIONS

Let us consider the two-dimensional homogenous wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial t^2} \tag{4}$$

We expand the solution $u = u(x, y, t)$ of (3) into a Taylor series in the neighbourhood of an arbitrary point (x_0, y_0, t_0)

$$u(x,y,t)=u(x_0,y_0,t_0)+ \\ +\sum_{n=1}^{\infty}\frac{d^n u(x_0,y_0,t_0)}{n!}(x-x_0,y-y_0,t-t_0) \tag{5}$$

where

$$d^n u(x_0,y_0,t_0)(x-x_0,y-y_0,t-t_0)= \\ =\left(\frac{\partial}{\partial x}(x-x_0)+\frac{\partial}{\partial y}(y-y_0)+\frac{\partial}{\partial t}(t-t_0)\right)^n u(x_0,y_0,t_0).$$

Using (4) in order to eliminate $\partial^2 u/\partial t^2$, we transform the right hand side of (5), and then group other partial derivatives in a specific way (it is essential to separate Δu). This leads to the following form of the expansion

$$u(x,y,t)=u_0+\frac{\partial u}{\partial x}\bar{x}+\frac{\partial u}{\partial y}\bar{y}+\frac{\partial u}{\partial t}\bar{t}+\frac{\partial^2 u}{\partial x^2}\left(\frac{\bar{x}^2}{2!}-\frac{\bar{y}^2}{2!}\right)+ \\ +\frac{\partial^2 u}{\partial x\partial y}\bar{x}\bar{y}+\frac{\partial^2 u}{\partial x\partial t}\bar{x}\bar{t}+\frac{\partial^2 u}{\partial y\partial t}\bar{y}\bar{t}+\Delta u\left(\frac{\bar{t}^2}{2!}+\frac{\bar{y}^2}{2!}\right)+ \\ +\frac{\partial^3 u}{\partial x^3}\left(\frac{\bar{x}^3}{3!}-\frac{\bar{x}\bar{y}^2}{2!}\right)+\frac{\partial^3 u}{\partial x^2\partial y}\left(\frac{\bar{x}^2\bar{y}}{2!}-\frac{\bar{y}^3}{3!}\right)+ \\ +\frac{\partial^3}{\partial x^2\partial t}\bar{t}\left(\frac{\bar{x}^2}{2!}-\frac{\bar{y}^2}{2!}\right)+\frac{\partial^3 u}{\partial x\partial y\partial t}\bar{t}\,\bar{x}\,\bar{y}+\frac{\partial}{\partial t}\Delta u\left(\frac{\bar{t}^3}{3!}+\bar{t}\frac{\bar{y}^2}{2!}\right)+ \\ +\frac{\partial}{\partial x}\Delta u\left(\frac{\bar{t}^2}{2!}\bar{x}+\frac{\bar{x}\bar{y}^2}{2!}\right)+\frac{\partial}{\partial y}\Delta u\left(\frac{\bar{t}^2}{2!}\bar{y}+\frac{\bar{y}^3}{3!}\right)+\ldots \tag{6}$$

where $u_0 = u(x_0, y_0, z_0)$, $\bar{x}=x-x_0$, $\bar{y}=y-y_0$, $\bar{t}=t-t_0$.

In the above-presented expansion partial derivatives $\frac{\partial^n u}{\partial x^n}$, $\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial u}{\partial y}\right)$ at (x_0, y_0, t_0) are multiplied by well known harmonic functions (7, 8) whereas other derivatives are multiplied by the polynomials satisfying Eq. (4).

$$F_n(x,y)=\mathrm{Re}\left(\frac{(x+iy)^n}{n!}\right)=\sum_{k=0,2,\ldots}^{n\ge k}(-1)^{\frac{k}{2}}\frac{x^{n-k}}{(n-k)!}\frac{y^k}{k!}\quad n\ge 0 \tag{7}$$

$$G_n(x,y)=\mathrm{Im}\left(\frac{(x+iy)^n}{n!}\right)=\sum_{k=1,3,\ldots}^{n\ge k}(-1)^{\frac{k-1}{2}}\frac{x^{n-k}}{(n-k)!}\frac{y^k}{k!}\quad n\ge 1 \tag{8}$$

All coefficients in (6) are Trefftz functions (wave polynomials) for Eq. (4). Moreover, they are different from wave polynomials described in [3]. In the next part we introduce the inverse Laplace operator for harmonic functions. It will be used to get wave polynomials appearing in (6).

IV. AN INVERSE LAPLACE OPERATOR FOR HARMONIC FUNCTIONS

On the analogy of [1, 7], we take into account a differential operator $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$.

Basis harmonic functions $F_n(x, y)$, $G_n(x, y)$ are linear combinations of monomials $(x^m/m!)(y^k/k!)$; to calculate their inverse Laplace operators it is enough to know $\Delta^{-n}(x^m/m!)(y^k/k!)$.

Since

$$\Delta\left(\frac{x^m}{m!}\frac{y^k}{k!}\right)= \\ =\frac{x^{m-2}}{(m-2)!}\frac{y^k}{k!}+\frac{x^m}{m!}\frac{y^{k-2}}{(k-2)!} \tag{9}$$

we may write

$$\Delta^{-1}\left(\frac{x^m}{m!}\frac{y^{k-2}}{(k-2)!}\right)= \\ =\frac{x^m}{m!}\frac{y^k}{k!}-\Delta^{-1}\left(\frac{x^{m-2}}{(m-2)!}\frac{y^k}{k!}\right) \tag{10}$$

$$m\ge 2, k\ge 2.$$

After substituting $k+2$ in place of k we obtain the relation

$$\Delta^{-1}\left(\frac{x^m}{m!}\frac{y^k}{(k)!}\right)= \\ =\frac{x^m}{m!}\frac{y^{k+2}}{(k+2)!}-\Delta^{-1}\left(\frac{x^{m-2}}{(m-2)!}\frac{y^{k+2}}{(k+2)!}\right) \tag{11}$$

$$m\ge 2,\ k\ge 0.$$

If we put

$$\Delta^{-1}\left(\frac{y^k}{k!}\right)=\frac{y^{k+2}}{(k+2)!}, \\ \Delta^{-1}\left(x\frac{y^k}{k!}\right)=x\frac{y^{k+2}}{(k+2)!} \tag{12}$$

then from (11), (12) we compute the inverse Laplace operator for an arbitrary monomial (13)

$$\Delta^{-1}\left(\frac{x^m}{m!}\frac{y^k}{k!}\right)=\begin{cases}\dfrac{x^m}{m!}\dfrac{y^{k+2}}{(k+2)!} & m=0,\ m=1,\ k\geq 0\\ \dfrac{x^m}{m!}\dfrac{y^{k+2}}{(k+2)!}-\Delta^{-1}\left(\dfrac{x^{m-2}}{(m-2)!}\dfrac{y^{k+2}}{(k+2)!}\right) & m\geq 2,\ k\geq 0\end{cases} \tag{13}$$

The formula (13) may be written in a closed form

$$\Delta^{-1}\left(\frac{x^m}{m!}\frac{y^k}{k!}\right)=\sum_{j=0}^{\left[\frac{m}{2}\right]}(-1)^j\frac{x^{m-2j}}{(m-2j)!}\frac{y^{k+2+2j}}{(k+2+2j)!} \tag{14}$$

Harmonic functions F_n, G_n are symmetrical with respect to variables x and y, whereas calculations of successive inverse operations in accordance with the formula (11) distinguish variable y (observe that $\Delta^{-1}(1)=y^2/2!$). By symmetry of Δ with respect to variables x and y, it is possible to define Δ^{-1} which distinguishes variable x. Both possibilities are shown in Table 1.

Table 1. Dependence of inverse operations of monomials with respect to distinguished variables

Distinguished variable	Inverse operation $\Delta^{-1}\left(\frac{x^m}{m!}\frac{y^k}{k!}\right)$
x $\quad\left(\Delta^{-1}(1)=\frac{x^2}{2!}\right)$	$\Delta^{-1}\left(\frac{x^m}{m!}\frac{y^k}{k!}\right)=\begin{cases}\frac{x^{m+2}}{(m+2)!}\frac{y^k}{k!} & k=0, k=1, m\geq 0\\ -\Delta^{-1}\left(\frac{x^{m+2}}{(m+2)!}\frac{y^{k-2}}{(k-2)!}\right)+\frac{x^{m+2}}{(m+2)!}\frac{y^k}{k!} & k\geq 2, m\geq 0\end{cases}$
y $\quad\left(\Delta^{-1}(1)=\frac{y^2}{2!}\right)$	$\Delta^{-1}\left(\frac{x^m}{m!}\cdot\frac{y^k}{k!}\right)=\begin{cases}\frac{x^m}{m!}\cdot\frac{y^{k+2}}{(k+2)!} & m=0,\ m=1,\ k\geq 0\\ -\Delta^{-1}\left(\frac{x^{m-2}}{(m-2)!}\cdot\frac{y^{k+2}}{(k+2)!}\right)+\frac{x^m}{m!}\cdot\frac{y^{k+2}}{(k+2)!} & m\geq 2,\ k\geq 0\end{cases}$

Similarly, the following inverse operations are calculated

$$\Delta^{-(n+1)}=\Delta^{-1}(\Delta^{-n}). \tag{15}$$

The inverse operations for a few successive harmonic functions $F_n(x,y)$ and $G_n(x,y)$ are shown below.

$$\begin{array}{lll}\Delta^{-1}(F_0)=\Delta^{-1}(1)=\dfrac{y^2}{2!} & \Delta^{-2}(F_0)=\Delta^{-1}\left(\dfrac{y^2}{2!}\right)=\dfrac{y^4}{4!} & \dots\\ \Delta^{-1}(G_1)=\Delta^{-1}(x)=\dfrac{xy^2}{2!} & \Delta^{-2}(G_1)=\Delta^{-1}\left(\dfrac{xy^2}{2!}\right)=\dfrac{xy^4}{4!} & \dots\\ \Delta^{-1}(F_1)=\Delta^{-1}(y)=\dfrac{y^3}{3!} & \Delta^{-2}(F_1)=\Delta^{-1}\left(\dfrac{y^3}{3!}\right)=\dfrac{y^5}{5!} & \dots\\ \Delta^{-1}(G_2)=\Delta^{-1}(xy)=\dfrac{xy^3}{3!} & \Delta^{-2}(G_2)=\Delta^{-1}\left(\dfrac{xy^3}{3!}\right)=\dfrac{xy^5}{5!} & \dots\end{array} \tag{16}$$

$$\Delta^{-1}(F_2)=\Delta^{-1}\left(\frac{x^2}{2!}-\frac{y^2}{2!}\right)=\frac{x^2y^2}{2!2!}-2\frac{y^4}{4!} \qquad \Delta^{-2}(F_2)=\Delta^{-1}\left(\frac{x^2y^2}{2!2!}-2\frac{y^4}{4!}\right)=\frac{x^2y^4}{2!4!}-3\frac{y^6}{6!}$$

...

V. REPRESENTATION OF WAVE FUNCTIONS IN TERMS OF INVERSE LAPLACE OPERATIONS

Let us divide the Taylor series expansion (6) into two parts (17). The first one includes harmonic functions F_n and functions connected with them by inverse operations, while the second one contains their Laplace inverse apart from harmonic functions G_n

$$\begin{aligned}
u(x,y,t) &= u_0 1 + \\
&+\frac{\partial u}{\partial x}\bar{x}+\frac{\partial u}{\partial t}\bar{t}+ \\
&+\frac{\partial^2 u}{\partial x^2}\left(\frac{\bar{x}^2}{2!}-\frac{\bar{y}^2}{2!}\right)+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)\bar{t}\bar{x}+\Delta u\left(\frac{\bar{t}^2}{2!}+\frac{\bar{y}^2}{2!}\right)+ \\
&+\frac{\partial^3 u}{\partial x^3}\left(\frac{\bar{x}^3}{3!}-\frac{\bar{x}\bar{y}^2}{2!}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)\bar{t}\left(\frac{\bar{x}^2}{2!}-\frac{\bar{y}^2}{2!}\right)+\frac{\partial}{\partial x}\Delta u\left(\frac{\bar{t}^2}{2!}\bar{x}+\bar{x}\frac{\bar{y}^2}{2!}\right)+\Delta\left(\frac{\partial u}{\partial t}\right)\left(\frac{\bar{t}^3}{3!}+\bar{t}\frac{\bar{y}^2}{2!}\right)+ \\
&+\frac{\partial^4 u}{\partial x^4}\left(\frac{\bar{x}^4}{4!}-\frac{\bar{x}^2\bar{y}^2}{2!2!}+\frac{\bar{y}^2}{4!}\right)+\frac{\partial^2}{\partial x^2}\Delta u\left(\frac{\bar{t}^2}{2!}\left(\frac{\bar{x}^2}{2!}-\frac{\bar{y}^2}{2!}\right)+\frac{\bar{x}^2\bar{y}^2}{2!2!}-2\frac{\bar{y}^4}{4!}\right)+\Delta\Delta u\left(\frac{\bar{t}^4}{4!}+\frac{\bar{t}^2}{2!}\frac{\bar{y}^2}{2!}+\frac{\bar{y}^4}{4!}\right)+\ldots \\
&+\frac{\partial u}{\partial y}\bar{y}+ \\
&+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\bar{x}\bar{y}+\frac{\partial^2 u}{\partial t\partial y}\bar{t}\bar{y}+ \\
&+\frac{\partial^3}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)\left(\frac{\bar{x}^2\bar{y}}{2!}-\frac{\bar{y}^3}{3!}\right)+\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t\partial y}\right)\bar{t}\bar{x}\bar{y}+\Delta\frac{\partial u}{\partial y}\left(\frac{\bar{t}^2}{2!}\bar{y}+\frac{\bar{y}^3}{3!}\right)+ \\
&+\frac{\partial^4}{\partial x^3}\left(\frac{\partial u}{\partial y}\right)\left(\frac{\bar{x}^3\bar{y}}{3!}-\frac{\bar{x}\bar{y}^3}{3!}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial t\partial y}\right)\bar{t}\left(\frac{\bar{x}^2}{2!}-\frac{\bar{y}^3}{3!}\right)+\frac{\partial}{\partial x}\left(\Delta\frac{\partial u}{\partial y}\right)\left(\frac{\bar{t}^2}{2!}\bar{x}\bar{y}+\frac{\bar{x}\bar{y}^3}{3!}\right)+\Delta\left(\frac{\partial^2 u}{\partial t\partial y}\right)\left(\frac{\bar{t}^3}{3!}\bar{y}+\bar{t}\frac{\bar{y}^3}{3!}\right)+\ldots
\end{aligned}\tag{17}$$

Using $\Delta^{-i}(F_k),\ \Delta^{-i}(G_k),\ i=1,2\ldots N,\ k=0,1,2\ldots$ allows us to rewrite formula (17)

$$\begin{aligned}
u(x,y,t) &= u_0 F_0 + \\
&+\frac{\partial u}{\partial x}F_1+\frac{\partial u}{\partial t}\bar{t}F_0+ \\
&+\frac{\partial^2 u}{\partial x^2}F_2+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)\bar{t}F_1+\Delta u\left(\frac{\bar{t}^2}{2!}F_0+\Delta^{-1}F_0\right)+ \\
&+\frac{\partial^3 u}{\partial x^3}F_3+\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)\bar{t}F_2+\frac{\partial}{\partial x}\Delta u\left(\frac{\bar{t}^2}{2!}F_1+\Delta^{-1}F_1\right)+\Delta u\left(\frac{\partial u}{\partial t}\right)\left(\frac{\bar{t}^3}{3!}F_0+\bar{t}\,\Delta^{-1}F_0\right)+ \\
&+\frac{\partial^4 u}{\partial x^4}F_4+\frac{\partial^3}{\partial x^3}\left(\frac{\partial u}{\partial t}\right)\bar{t}F_3+\frac{\partial^2}{\partial x^2}\Delta u\left(\frac{\bar{t}^2}{2!}F_2+\Delta^{-1}F_2\right)+\frac{\partial}{\partial x}\left(\Delta\frac{\partial u}{\partial t}\right)\left(\frac{\bar{t}^3}{3!}F_1+\bar{t}\Delta^{-1}F_1\right)+\Delta\Delta u\left(\frac{\bar{t}^4}{4!}F_0+\frac{\bar{t}^2}{2!}\Delta^{-1}F_0+\Delta^{-2}F_0\right)+\ldots
\end{aligned}\tag{18}$$

$$+\frac{\partial u}{\partial y}G_1+$$

$$+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)G_2+\frac{\partial^2 u}{\partial t\partial y}\bar{t}G_1+$$

$$+\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)G_3+\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t\partial y}\right)\bar{t}G_2+\Delta\left(\frac{\partial u}{\partial y}\right)\left(\frac{\bar{t}^2}{2!}G_1+\Delta^{-1}G_1\right)+$$

$$+\frac{\partial^3}{\partial x^3}\left(\frac{\partial u}{\partial y}\right)G_4+\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial t\partial y}\right)\bar{t}G_3+\frac{\partial}{\partial x}\Delta\left(\frac{\partial u}{\partial y}\right)\left(\frac{\bar{t}^2}{2!}G_2+\Delta^{-1}G_2\right)+\Delta u\left(\frac{\partial^2 u}{\partial t\partial y}\right)\left(\frac{\bar{t}^3}{3!}G_1+\bar{t}\,\Delta^{-1}G_1\right)+$$

$$+\frac{\partial^4}{\partial x^4}\left(\frac{\partial u}{\partial y}\right)G_5+\frac{\partial^3}{\partial x^3}\left(\frac{\partial^2 u}{\partial t\partial y}\right)\bar{t}G_4+\frac{\partial^2}{\partial x^2}\Delta\left(\frac{\partial u}{\partial y}\right)\left(\frac{\bar{t}^2}{2!}G_3+\Delta^{-1}G_3\right)+\frac{\partial}{\partial x}\Delta\left(\frac{\partial^2 u}{\partial t\partial y}\right)\left(\frac{\bar{t}^3}{3!}G_2+\bar{t}\Delta^{-1}G_2\right)+\ldots$$

$$=\sum_{n=0}^{N}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\partial^{n-2k}}{\partial x^{n-2k}}(\Delta^k u)\left(\sum_{j=0}^{k}\frac{\bar{t}^{2j}}{(2j)!}\Delta^{-k+j}F_{n-2k}(x,y)\right)+\sum_{n=1}^{N}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\partial^{n-1-2k}}{\partial x^{n-1-2k}}\left(\Delta^k\frac{\partial u}{\partial t}\right)\left(\sum_{j=0}^{k}\frac{\bar{t}^{2j+1}}{(2j+1)!}\Delta^{-k+j}F_{n-1-2k}(x,y)\right)+$$

$$+\sum_{n=1}^{N}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\partial^{n-1-2k}}{\partial x^{n-1-2k}}\left(\Delta^k\frac{\partial u}{\partial y}\right)\left(\sum_{j=0}^{k}\frac{\bar{t}^{2j}}{(2j)!}\Delta^{-k+j}G_{n-2k}(x,y)\right)+\sum_{n=2}^{N}\sum_{k=0}^{\left[\frac{n-2}{2}\right]}\frac{\partial^{n-2-2k}}{\partial x^{n-2-2k}}\left(\Delta^k\frac{\partial u}{\partial t\partial y}\right)\left(\sum_{j=0}^{k}\frac{\bar{t}^{2j+1}}{(2j+1)!}\Delta^{-k+j}G_{n-1-2k}(x,y)\right)+R_{N+1},$$

where $[x]$ gives the greatest integer less than or equal to x.

The obtained form of *u(x,y,t)* consists of two independent sequences of wave polynomials given by inverse operations of harmonic functions.

We can repeat all the previous considerations to eliminate the partial derivatives from the Taylor series expansion with the help of the wave equation. Then we obtain a similar form of function u, but now the inverse operations are calculated in a different way – with distinguished different variables. The results of all possible eliminations and consecutive expansions are presented in Table 2.

Table 2. Relationship of expanded form of $u(x, y, t)$ with the inverse Laplace operator and eliminated derivative

Eliminated derivatives	Distinguished variable in inverse operations	Expanded form of $u = u(x, y, t)$
$\frac{\partial^2 u}{\partial x^2}$	x $\left(\Delta^{-1}(1)=\frac{x^2}{2!}\right)$	$u(x,y,t)=\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{\partial^n u}{\partial y^k\partial t^{n-k}}\left(\sum_{j=0}^{\left[\frac{n-k}{2}\right]}\frac{t^{n-k-2j}}{(n-k-2j)!}\Delta^{-j}F_k(y,x)\right)+$ $+\sum_{n=1}^{N}\sum_{k=0}^{n-1}\frac{\partial^n u}{\partial y^k\partial t^{n-1-k}\partial x}\left(\sum_{j=0}^{\left[\frac{n-k-1}{2}\right]}\frac{t^{n-k-1-2j}}{(n-k-1-2j)!}\Delta^{-j}G_{k+1}(y,x)\right)+R_{N+1}$
$\frac{\partial^2 u}{\partial y^2}$	y $\left(\Delta^{-1}(1)=\frac{y^2}{2!}\right)$	$u(x,y,t)=\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{\partial^n u}{\partial x^k\partial t^{n-k}}\left(\sum_{j=0}^{\left[\frac{n-k}{2}\right]}\frac{t^{n-k-2j}}{(n-k-2j)!}\Delta^{-j}F_k(x,y)\right)+$ $+\sum_{n=1}^{N}\sum_{k=0}^{n-1}\frac{\partial^n u}{\partial x^k\partial t^{n-1-k}\partial y}\left(\sum_{j=0}^{r\left[\frac{n-k-1}{2}\right]}\frac{t^{n-k-1-2j}}{(n-k-1-2j)!}\Delta^{-j}G_{k+1}(x,y)\right)+R_{N+1}$

$\dfrac{\partial^2 u}{\partial t^2}$	y $\left(\Delta^{-1}(1)=\dfrac{y^2}{2!}\right)$	$\begin{aligned} u(x,y,t) &= \sum_{n=0}^{N}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\partial^{n-2k}}{\partial x^{n-2k}}(\Delta^k u)\left(\sum_{j=0}^{k}\frac{t^{2j}}{(2j)!}\Delta^{-k+j}F_{n-2k}(x,y)\right)+ \\ &+\sum_{n=1}^{N}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\partial^{n-1-2k}}{\partial x^{n-1-2k}}\left(\Delta^k\frac{\partial u}{\partial t}\right)\left(\sum_{j=0}^{k}\frac{t^{2j+1}}{(2j+1)!}\Delta^{-k+j}F_{n-1-2k}(x,y)\right)+ \\ &+\sum_{n=1}^{N}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\partial^{n-1-2k}}{\partial x^{n-1-2k}}\left(\Delta^k\frac{\partial u}{\partial y}\right)\left(\sum_{j=0}^{k}\frac{t^{2j}}{(2j)!}\Delta^{-k+j}G_{n-2k}(x,y)\right)+ \\ &+\sum_{n=2}^{N}\sum_{k=0}^{r\left[\frac{n-2}{2}\right]}\frac{\partial^{n-2-2k}}{\partial x^{n-2-2k}}\left(\Delta^k\frac{\partial u}{\partial t\partial y}\right)\left(\sum_{j=0}^{k}\frac{t^{2j+1}}{(2j+1)!}\Delta^{-k+j}G_{n-1-2k}(x,y)\right)+R_{N+1} \end{aligned}$

VI. STRUCTURE OF AN APPROXIMATE SOLUTION OF THE WAVE EQUATION

The solution of the wave equation is approximated by the sum of two linear combinations of wave polynomials

$$\Theta(x,\ y,\ t)\approx\sum_{n=0}^{N}a_n\cdot\varphi_n(x,y,t)+\sum_{k=1}^{N}a_k\psi_k(x,y,t) \tag{19}$$

where $\varphi_n(x,y,t)$, $\psi_k(x,y,t)$ denote functions given by inverse operations of harmonic functions F_n, G_k respectively, N – degree of harmonic functions.

Denote $h_{2n}(x,y)=F_n(x,y),\ h_{2n+1}(x,y)=G_{n+1}(x,y),\ n\ge 0$ and arrange wave polynomials $\varphi_n(x,y,t)$, $\psi_k(x,y,t)$ into a triangle matrix (20). In every row there are polynomials of the fixed degree with respect to variables $x,\ y$. Each column contains polynomials of the same degree with respect to variable t.

$$\begin{array}{lllll}
\varphi_0=h_0 & & & & \\
\phi_1=h_1 & & & & \\
\varphi_1=h_2 & \varphi_2=th_0 & & & \\
\phi_2=h_3 & \phi_3=th_1 & & & \\
\varphi_3=h_4 & \varphi_4=th_2 & \varphi_5=\dfrac{t^2}{2!}h_0+\Delta^{-1}h_0 & & \\
\phi_4=h_5 & \phi_5=th_3 & \phi_6=\dfrac{t^2}{2!}h_1+\Delta^{-1}h_1 & & \\
\varphi_6=h_6 & \varphi_7=th_4 & \varphi_8=\dfrac{t^2}{2!}h_2+\Delta^{-1}h_2 & \varphi_9=\dfrac{t^3}{3!}h_0+t\,\Delta^{-1}h_0 & \\
\phi_7=h_7 & \phi_8=th_5 & \phi_9=\dfrac{t^2}{2!}h_3+\Delta^{-1}h_3 & \phi_{10}=\dfrac{t^3}{3!}h_1+t\,\Delta^{-1}h_1 & \\
\varphi_{10}=h_8 & \varphi_{11}=th_6 & \varphi_{12}=\dfrac{t^2}{2!}h_4+\Delta^{-1}h_4 & \varphi_{13}=\dfrac{t^3}{3!}h_2+t\,\Delta^{-1}h_2 & \varphi_{14}=\dfrac{t^4}{4!}h_0+\dfrac{t^2}{2!}\Delta^{-1}h_0+\Delta^{-2}h_0.
\end{array} \tag{20}$$

For an approximation purpose we always require a finite number of functions, so the question is in which way and how many functions we should choose. Both the degree of harmonic functions and the assumption of accuracy with respect to the time variable determine the unambiguous choice of functions presented in Table 3.

Table 3. Elements of linear combinations of base functions depending on order approximations with respect to time

Order of polynomials with respect to space variables	$\sim t^0$	$\sim t^1$	$\sim t^2$	$\sim t^3$	$\sim t^4$	$\sim t^5$
	Harmonic base function h_i	Harmonic base function h_i	Biharmonic base function $\Delta^{-1} h_i$	Biharmonic base function $\Delta^{-1} h_i$	Base function $\Delta^{-2} h_i$	Base function $\Delta^{-2} h_i$
0	h_0					
1	h_1 h_2	h_0				
2	h_3 h_4	h_1 h_2	$\Delta^{-1} h_0$			
3	h_5 h_6	h_3 h_4	$\Delta^{-1} h_1$ $\Delta^{-1} h_2$	$\Delta^{-1} h_0$		
4	h_7 h_8	h_5 h_6	$\Delta^{-1} h_3$ $\Delta^{-1} h_4$	$\Delta^{-1} h_1$ $\Delta^{-1} \mathrm{h}_2$	$\Delta^{-2} h_0$	
5	h_9 h_{10}	h_7 h_8	$\Delta^{-1} h_5$ $\Delta^{-1} h_6$	$\Delta^{-1} h_3$ $\Delta^{-1} h_4$	$\Delta^{-2} h_1$ $\Delta^{-2} h_2$	$\Delta^{-2} h_0$
6	h_{11} h_{12}	h_9 h_{10}	$\Delta^{-1} h_7$ $\Delta^{-1} h_8$	$\Delta^{-1} h_5$ $\Delta^{-1} h_6$	$\Delta^{-2} h_3$ $\Delta^{-2} h_4$	$\Delta^{-2} h_1$ $\Delta^{-2} h_2$
7	h_{13} h_{14}	h_{11} h_{12}	$\Delta^{-1} h_9$ $\Delta^{-1} h_{10}$	$\Delta^{-1} h_7$ $\Delta^{-1} h_8$	$\Delta^{-2} h_5$ $\Delta^{-2} h_6$	$\Delta^{-2} h_3$ $\Delta^{-2} h_4$
8	h_{15} h_{16}	h_{13} h_{14}	$\Delta^{-1} h_{11}$ $\Delta^{-1} h_{12}$	$\Delta^{-1} h_9$ $\Delta^{-1} h_{10}$	$\Delta^{-2} h_7$ $\Delta^{-2} h_8$	$\Delta^{-2} h_5$ $\Delta^{-2} h_6$

The hypothetical system of approximating functions is shown below.

The linear combination of base functions, which gives polynomials up to degree 1 with respect to time and up to degree 3 with respect to space variables consists of 12 functions

$$h_0,\ h_1,\ h_2,\ h_3,\ h_4,\ h_5,\ h_6$$
$$th_0,\ \ h_1,\ th_2, th_3,\ th_4.$$

If we require the quadratic approximation with respect to t and the approximation up to degree 4 with respect to x and y, we should take 25 functions

$$h_0,\ h_1,\ h_2,\ h_3,\ h_4,\ h_5,\ h_6,\ h_7,\ h_8$$
$$t\,h_0,\ t\,h_1,\ t\,h_2, t\,h_3,\ t\,h_4,\ t\,h_5,\ t\,h_6$$
$$\frac{t^2}{2!}h_0+\Delta^{-1}h_0,\ \frac{t^2}{2!}h_1+\Delta^{-1}h_1,\ \frac{t^2}{2!}h_2+\Delta^{-1}h_2,\ \frac{t^2}{2!}h_3+\Delta^{-1}h_3,\ \frac{t^2}{2!}h_4+\Delta^{-1}h_4,$$
$$\frac{t^3}{3!}h_0+t\,\Delta^{-1}h_0,\ \frac{t^3}{3!}h_1+t\,\Delta^{-1}h_1,\ \frac{t^3}{3!}h_2+t\,\Delta^{-1}h_2$$
$$\frac{t^4}{4!}h_0+\frac{t^2}{2!}\Delta^{-1}h_0+t\,\Delta^{-2}h_0.$$

Generally, if an approximation of an exact solution consists of polynomials up to degree n with respect to x and y and up to degree m with respect to t, functions used for the approximation are chosen in the following way. From the column with number j ($1 \le j \le m - 1$) in Table 3, $2(n - j + 1)$ consecutive functions should be taken. Notice that column 1 includes only harmonic functions h_i, whereas column j includes components $\Delta^{-[(j-1)/n]}h_i$. When we omit a restriction on the order of time, the approximation is exactly the truncated Taylor expansion of $u = u(x, y, t)$ at (x_0, y_0, t_0) up to the order n.

VII. POLAR WAVE FUNCTIONS

In polar coordinates the two-dimensional wave equation has the following form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}. \tag{21}$$

Like in Cartesian coordinates, the solution of a problem given by equation (21) with appropriate initial and boundary conditions can be approximated by a linear combination of polar wave functions (23) satisfying the governing equation. These functions are obtained by substituting

$$x = r\cos\varphi, \quad y = r\sin\varphi \tag{22}$$

into (20).

A few of such functions are presented below:

$$\begin{array}{lll}
h_0 = 1, & & \\
h_1 = r\sin\varphi, & & \\
h_2 = r\cos\varphi & th_0 = t, & \\
h_3 = r^2\sin\varphi\cos\varphi, & th_1 = tr\sin\varphi & \frac{t^2}{2!}h_0 + \Delta^{-1}h_0 = \frac{t^2}{2!} + \frac{r^2\sin^2\varphi}{2!} \\
h_4 = \frac{r^2}{2!}(-\cos 2\varphi), & th_2 = tr\cos\varphi & \frac{t^2}{2!}h_1 + \Delta^{-1}h_1 = \frac{t^2}{2!}r\sin\varphi + \frac{r^3\sin^3\varphi}{3!} \\
h_5 = r^2\sin\varphi\cos\varphi, & th_3 = tr\sin\varphi, & \frac{t^2}{2!}h_2 + \Delta^{-1}h_2 = \frac{t^2}{2!}r\cos\varphi + \frac{r^3\cos\varphi\sin^2\varphi}{2!}
\end{array} \tag{23}$$

An approximate solution is formed exactly in the same way as in Cartesian coordinates.

VIII. NUMERICAL EXAMPLES

The authors of papers [1] and [3] have shown how the wave-polynomial method works for polynomials generated for one and two-dimensional Cartesian coordinates. Numerical examples presented here demonstrate how this method works for wave polynomials given by inverse Laplace operators.

- **Vibrations of a square membrane**

The two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (x, y) \in (0,1)\times(0,1), \; t \ge 0 \tag{24}$$

together with conditions:

$$\begin{array}{l} u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0 \\ u(x, y, 0) = xy(x-1)(y-1) \quad \frac{\partial u}{\partial t}(x, y, 0) = 0 \end{array} \tag{25}$$

has the exact solution given by [5]

$$u(x, y, t) = \frac{64}{\pi^6}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\sin(\pi x(2n+1))\sin(\pi y(2m+1))\cos\left(\pi t\sqrt{(2n+1)^2 + (2m+1)^2}\right)}{(2n+1)^3(2m+1)^3}. \tag{26}$$

An approximate solution of this problem $\Theta(x, y, t)$ is determined in a successive manner in the n-th time interval $\langle (n-1)\delta t,\ n\delta t \rangle$ as a linear combination of the Trefftz functions

$$\Theta_n(x,y,t) = \sum_{k=0}^{N} a_k h_k(x,y,t). \tag{27}$$

The unknown coefficients of this approximation a_k are obtained from the minimization of the following functional

$$\begin{aligned} J_n = & \int_0^1\int_0^1 \left(\Theta_n(x,y,(n-1)\delta t) - \Theta_{n-1}(x,y,(n-1)\delta t)\right)^2 dxdy + \\ & + \int_0^1\int_0^1 \left(\frac{\partial\Theta_n(x,y,(n-1)\delta t)}{\partial t} - \frac{\partial\Theta_{n-1}(x,y,(n-1)\delta t)}{\partial t}\right)^2 dxdy + \\ & + \int_0^1 \int_{(n-1)\delta t}^{n\delta t} \left(\Theta_n(x,0,t)\right)^2 dtdx + \int_0^1 \int_{(n-1)\delta t}^{n\delta t} \left(\Theta_n(x,1,t)\right)^2 dtdx + \\ & + \int_0^1 \int_{(n-1)\delta t}^{n\delta t} \left(\Theta_n(0,y,t)\right)^2 dtdy + \int_0^1 \int_{(n-1)\delta t}^{n\delta t} \left(\Theta_n(1,y,t)\right)^2 dtdy, \end{aligned} \tag{28}$$

which leads to a linear system of equations on a_k. We assume that in the first time interval, $\Theta_0(x, y, 0)$ is known from the initial conditions. In the n-th time interval $\langle (n-1)\delta t,\ n\delta t \rangle$ is given by the approximate solution at the end of the previous time interval.

To solve the considered problem we take an approximation by the Trefftz polynomials whose degrees are bounded by 10 with respect to space variables and 9 with respect to time.

Figure 1 shows the results obtained at time $t = 0.5$ for $\delta t = 1$.

Because the greatest deflection takes place in the middle of the membrane, Fig. 3 shows the results obtained in the whole time interval at this point.

The approximate solution obtained for a given time interval can be extended over this interval. Figure 4 shows that the extrapolation in time of the approximate solution is close to the exact one.

The extrapolation can obviously be made not only in time but also in space, so the method should give good results in solving inverse problems.

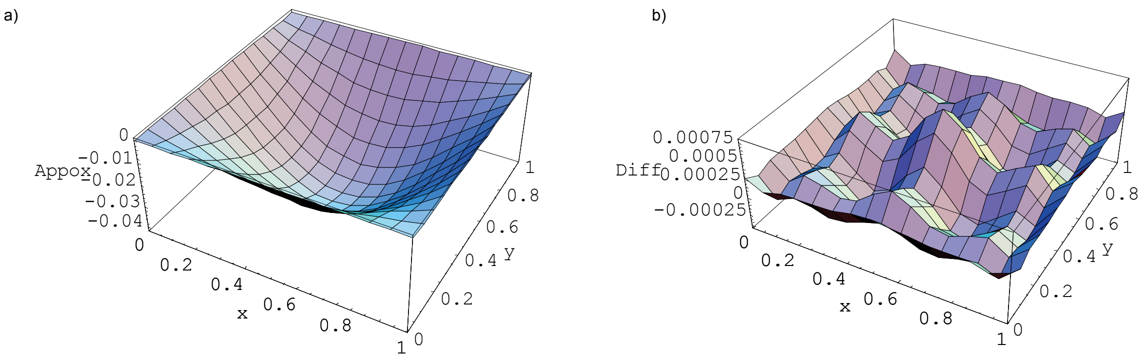

Fig. 1. Approximate solution (a), difference between exact and approximate solution (b) at $t = 0.5$

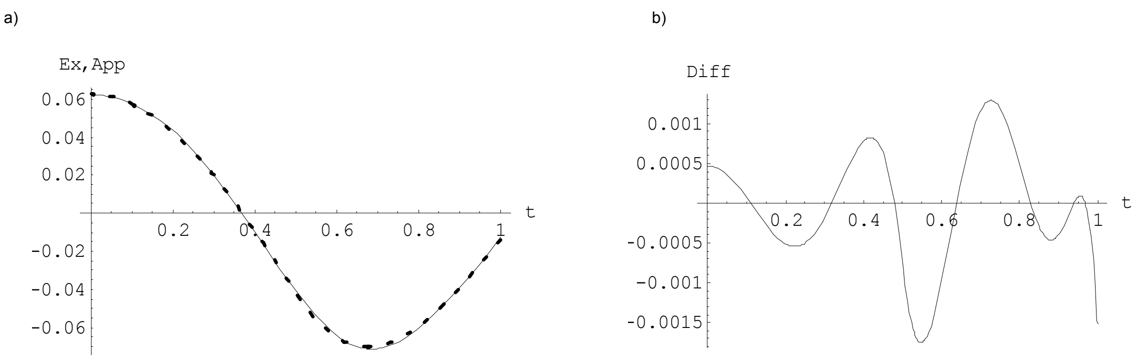

Fig. 2. Approximate and exact solution (a), difference between exact and approximate solution (b) at point (0.5, 0.5) for time interval <0, 1>

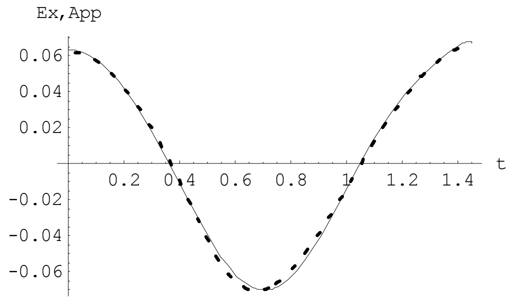

Fig. 3. Exact and approximate solu-tion in the middle of membrane for time interval <0, 1.5δt>

- **Vibrations of a square membrane with a constant source**

The next example shows the vibration of a membrane with the constant source $Q(x, y, t) = 0.1$. The equation of motion is given by

$$\frac{\partial u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Q(x, y, t) \quad (x, y) \in (0,1)\times(0,1), \qquad (29)$$

$$t > 0$$

with initial and boundary conditions:

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$$
$$u(x, y, 0) = 0 \quad \frac{\partial u}{\partial t}(x, y, 0) = 0 \qquad (30)$$

The exact solution for this problem is known [5]

$$u(x, y, t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{4\sin(k\pi x)\sin(l\pi y)}{10\sqrt{(k\pi)^2 + (l\pi)^2}} \times$$
$$\times \int_0^1\int_0^1\int_0^t \sin(k\pi x)\sin(l\pi y)\sin\left(\sqrt{(k\pi)^2 + (l\pi)^2}\,(t-\tau)\right) d\tau dy dx. \qquad (31)$$

As in the previous example the approximate solution to this problem $\Theta(x, y, t)$ is determined in a successive manner in the n-th time interval as a sum of linear combination of the Trefftz functions (27) and additionally function $g_p(x, y, t)$ which is the particular solution to the non-homogenous equation (29).

$$\Theta_n(x, y, t) = \sum_{k=0}^{N} a_k h_k(x, y, t) + g_p(x, y, t). \qquad (32)$$

The unknown coefficients of this approximation a_k are obtained from minimization of the functional (28), whereas function $g_p(x, y, t)$ is calculated as [3]

$$g_p = L^{-1}(Q) = L^{-1}\left(\sum_{n=0}^{\infty}\sum_{k+l+m=n} \frac{\partial^{(k+l+m)}Q(x_0, y_0, t_0)}{\partial x^k \partial y^l \partial t^m}\frac{\bar{x}^k\bar{y}^l\bar{t}^m}{k!l!m!}\right) =$$
$$= \sum_{n=0}^{\infty}\sum_{k+l+m=n} c_{klm} L^{-1}(\bar{x}^k\bar{y}^l\bar{t}^m), \qquad (33)$$

$$L^{-1}\left(\bar{x}^k\bar{y}^l\bar{t}^m\right) = \frac{\left(w_{p1} + w_{p2} + w_{p3}\right)}{3}, \qquad (34)$$

$$w_{p1} = L^{-1}(\bar{x}^k\bar{y}^l\bar{t}^m) = \frac{1}{(k+2)(k+1)} \times$$
$$\times (-\bar{x}^{k+2}\bar{y}^l\bar{t}^m + m(m-1)L^{-1}(\bar{x}^{k+2}\bar{y}^l\bar{t}^{m-2}) +$$
$$- l(l-1)L^{-1}(\bar{x}^{k+2}\bar{y}^{l-2}\bar{t}^m)),$$

$$w_{p2} = L^{-1}(\bar{x}^k\bar{y}^l\bar{t}^m) = \frac{1}{(l+2)(l+1)} \times$$
$$\times (-\bar{x}^k\bar{y}^{l+2}\bar{t}^m + m(m-1)L^{-1}(\bar{x}^k\bar{y}^{l+2}\bar{t}^{m-2}) + \qquad (35)$$
$$- k(k-1)L^{-1}(\bar{x}^{k-2}\bar{y}^{l+2}\bar{t}^m)),$$

$$w_{p3} = L^{-1}(\bar{x}^k\bar{y}^l\bar{t}^m) = \frac{1}{(m+2)(m+1)} \times$$
$$\times (-\bar{x}^k\bar{y}^l\bar{t}^{m+2} + k(k-1)L^{-1}(\bar{x}^{k+2}\bar{y}^l\bar{t}^{m-2}) +$$
$$- l(l-1)L^{-1}(\bar{x}^k\bar{y}^{l-2}\bar{t}^{m+2})),$$

where $\bar{x} = x - x_0, \bar{y} = y - y_0, \bar{t} = t - t_0$.

The same polynomials as in the previous example are used for the solution; function $g_p(x, y, t)$ is taken as

$$g_p(x, y, t) = \frac{-x^2 - y^2 + t^2}{60}. \qquad (36)$$

Figure 5 shows the results obtained at time $t = 0.5$ for $\delta t = 1$.

The approximate solution in the middle of the membrane and its extrapolation in time are shown in Fig. 6.

The error of approximation in the middle of the membrane defined as

$$\left(\frac{\int_0^{\delta t}\left(\Theta(0.5, 0.5, t) - u(0.5, 0.5, t)\right)^2 dt}{\int_0^{\delta t}\left(u(0.5, 0.5, t)\right)^2 dt}\right)^{1/2} \qquad (37)$$

is calculated. The results are shown in Table 4.

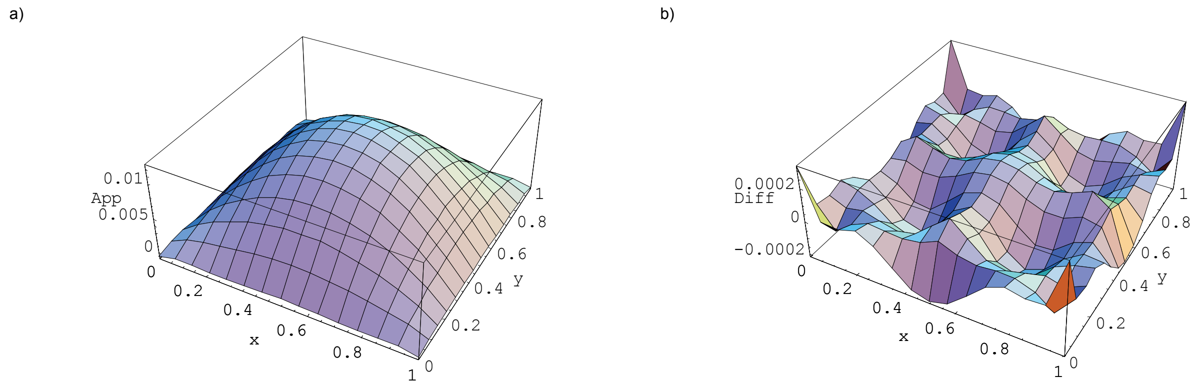

Fig. 4. Approximate solution (a), difference between exact and approximate solution (b) at $t = 0.5$

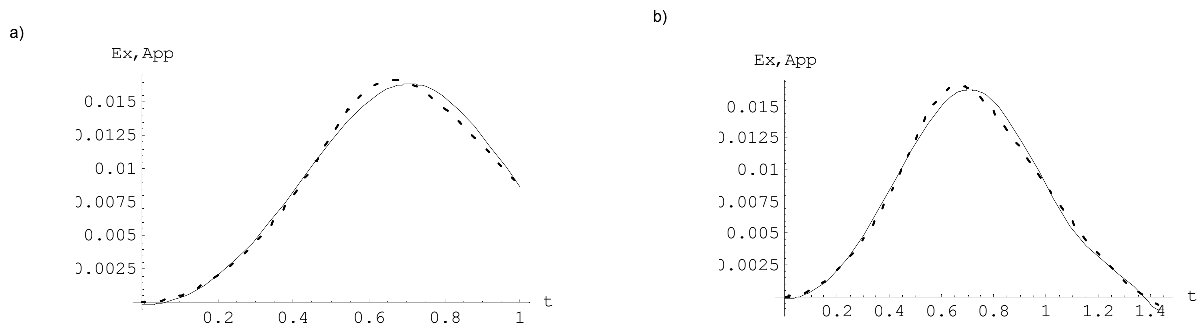

Fig. 5. Approximate and exact solution at point (0.5, 0.5, 1) (a). Approximate and exact solution at point (0.5, 0.5, 1.5) (b)

Table 4. Error dependence on the number of polynomials and length of time interval

Number of polynomials	Length of time interval		
	$\delta t = 0.5$	$\delta t = 0.7$	$\delta t = 1$
$N = 49$	0.0070	0.0860	0.1769
$N = 85$	0.0565	0.1034	0.1612
$N = 121$	0.0141	0.0387	0.0556
$N = 153$	0.0200	0.0245	0.0458

The presented test example shows good accuracy the approximate solution with wave polynomials as base functions. Moreover, wave polynomials do not yield any restrictions on the shape of the area, so the method can be used for more complicated shapes of the considered areas. In these cases wave polynomials may also be used as base functions in different variants of FEM, which leads to better accuracy of approximation in each element.

The method of generating wave functions presented here can also be easily extended to a three-dimensional equation.

Acknowledgements

This work was carried out in the framework of the research project No N513 003 32/0541, which was financed from the resources for the development of science in the years 2007-2009.

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