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On Two Families of Implicit Interval Methods of Adams-Moulton Type

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Abstract: In our previous paper [1] we have presented implicit interval methods of Adams-Moulton type. It appears that two families of these types of methods exist. We compare both families of methods and present a numerical example.

Key words: initial value problem, interval methods, floating-point interval arithmetic

1. INTRODUCTION

In 1980s the explicit interval methods of Adams-Bashforth type has been first introduced by Šokin, Kalmykov and Juldašev (see *e.g.* [5, 10]). The research on the multistep interval methods has been continued by Marciniak and Jankowska [1-3, 7, 8]. The implicit interval methods of Adams-Moulton type have been presented in [1]. In this paper we continue our previous considerations and specify a precise form of two different families of implicit interval methods of this kind. This distinction follows from some interval arithmetic properties. It also depends on which of two conventionally equivalent formulas is chosen as the basis of its interval counterpart (see Sec. 2). In Sec. 3 we consider the equation of motion of simple pendulum and present some numerical results. A short summary given in Sec. 4 brings this paper to the end.

2. TWO FAMILIES OF IMPLICIT INTERVAL METHODS OF ADAMS-MOULTON TYPE

Let us consider the initial value problem (IVP) of the form

$$y' = f(t, y), \quad 0 \le t \le \xi, \qquad y(0) = y_0, \tag{1}$$

where

$$\xi \in \mathbf{R},\ y = y(t) \in \mathbf{R}^N, \quad f : [0, \xi] \times \mathbf{R}^N \to \mathbf{R}^N.$$

Let us choose a positive integer m and select the mesh points $t_i = ih$, $i = 0, 1, \ldots, m$, where $h = \xi/m$. It is well-known (see *e.g.* [6]) that there are two equivalent formulas that describe the exact solution of (1) at t_n, $n = k, k + 1, \ldots, m$. On the one hand we have

$$y(t_n) = y(t_{n-1}) + h\sum_{j=0}^{k} \overline{\gamma}_j \nabla^j f(t_n, y(t_n)) + h^{k+2}\overline{\gamma}_{k+1}\overline{\psi}(\eta, y(\eta)), \tag{2}$$

where

$$\nabla^j f(t_n, y(t_n)) = \sum_{m=0}^{j} (-1)^m \binom{j}{m} f(t_{n-m}, y(t_{n-m})), \tag{3}$$

$$\overline{\gamma}_0 = 1, \quad \overline{\gamma}_j = \frac{1}{j!}\int_{-1}^{0} s(s+1)\ldots(s+j-1)\,ds, \quad j = 1, 2, \ldots, k+1, \tag{4}$$

and

$$\bar{\psi}(\eta, y(\eta)) \equiv f^{(k+1)}(\eta, y(\eta)) \equiv y^{(k+2)}(\eta), \quad \eta \in [t_{n-k}, t_n]. \tag{5}$$

On the other hand, since

$$\bar{\beta}_{kj} = (-1)^j \sum_{m=j}^{k} \binom{m}{j} \bar{\gamma}_m, \quad j = 0, 1, \dots k, \tag{6}$$

we can write (2) in the form

$$y(t_n) = y(t_{n+1}) + h \sum_{j=0}^{k} \bar{\beta}_{kj} f(t_{n-j}, y(t_{n-j})) + h^{k+2} \bar{\gamma}_{k+1} \bar{\psi}(\eta, y(\eta)). \tag{7}$$

Let us denote

Δ_t, Δ_y – sets in which the function $f(t, y)$ is defined, *i.e.*

$$\Delta_t = \{t \in \mathbf{R} : 0 \le t \le \xi\},$$

$$\Delta_y = \Big\{y = (y_1, y_2, \dots, y_N)^T \in \mathbf{R}^N : \underline{b}_i \le y_i \le \bar{b}_i, \underline{b}_i, \bar{b}_i \in \mathbf{R}, \ i = 1, 2, \dots, N\Big\},$$

$F(T,Y)$ – interval extension of $f(t, y)$ (for the definition of interval extension see *e.g.* [4, 9-10]),

$\bar{\Psi}(T, Y)$ – interval extension of $\bar{\psi}(t, y)$ (see (5)).

Other assumptions about $F(T, Y)$ and $\bar{\Psi}(T, Y)$ are the same as in [1]. Furthermore, we denote by $I(.)$ the space of intervals over (.).

Now, let us assume that $y(0) \in Y_0$ and the intervals Y_i such that $y(t_i) \in Y_i$, $i = 1, 2, \dots, k-1$, are known. Then, the implicit interval methods of Adams-Moulton type can be defined as follows (see also [1]):

$$Y_n = Y_{n-1} + h \sum_{j=0}^{k} \bar{\gamma}_j \nabla^j F_n + h^{k+2} \bar{\gamma}_{k+1} \bar{\Psi}(T_n + [-kh, 0], Y_n + [-kh, 0] F(\Delta_t, \Delta_y)), \quad n = k, k+1, \dots, m, \tag{8}$$

where $h = \xi / m$, $t_i = ih \in T_i$, $i = 0, 1, \dots, m$, $\bar{\gamma}_j$, $j = 0, 1, \dots, k+1$, are given by (4), $F_n = F(T_n, Y_n)$ and

$$\nabla^j F_n = \sum_{m=0}^{j} (-1)^m \binom{j}{m} F_{n-m}.$$

Moreover, the Eq. (8) can be written in the equivalent form as follows:

$$Y_n = Y_{n-1} + h \sum_{j=0}^{k} \bar{\gamma}_j F_n + h \sum_{j=1}^{k} \bar{\gamma}_j \sum_{m=1}^{j} (-1)^m \binom{j}{m} F_{n-m} + h^{k+2} \bar{\gamma}_{k+1} \bar{\Psi}(T_n + [-kh, 0], Y_n + [-kh, 0] F(\Delta_t, \Delta_y)), \quad n = k, k+1, \dots, m. \tag{9}$$

In particular for a given k from (8) (or (9)) we get the following methods:

- $k = 1$

$$Y_n = Y_{n-1} + \frac{h}{2}(2F_n - F_n + F_{n-1}) + - \frac{h^3}{12} \bar{\Psi}(T_n + [-h, 0], \ Y_n + [-h, 0] F(\Delta_t, \Delta_y)),$$

- $k = 2$

$$Y_n = Y_{n-1} + \frac{h}{12}(12F_n - 7F_n + 8F_{n-1} - F_{n-2}) + - \frac{h^4}{24} \bar{\Psi}(T_n + [-2h, 0], \ Y_n + [-2h, 0] F(\Delta_t, \Delta_y)),$$

- $k = 3$

$$Y_n = Y_{n-1} + \frac{h}{24}(24F_n - 15F_n + 19F_{n-1} - 5F_{n-2} + F_{n-3}) + - \frac{19}{720} h^5 \bar{\Psi}(T_n + [-3h, 0], \ Y_n + [-3h, 0] F(\Delta_t, \Delta_y)).$$

Let us notice that for interval arithmetic the distributive law is not generally satisfied. However, since an interval is also a set, then for $X, Y, Z \in I(\mathbf{R})$, the following relation, as the subdistributive law, holds

$$X \cdot (Y + Z) \subset X \cdot Y + X \cdot Z. \tag{10}$$

Hence, the values of the interval extensions of f in the above formulas with the same indices cannot be subtracted.

In real arithmetic we have

$$\sum_{j=0}^{k} \bar{\beta}_{kj} f_{n-j} = \sum_{j=0}^{k} \bar{\gamma}_j \nabla^j f_n,$$

where $f_{n-j} = f(t_{n-j}, y(t_{n-j}))$, $j = 0, 1, \dots, k$. Hence, the formula (2) is equivalent to (7). But in interval arithmetic we have

$$\sum_{j=0}^{k} \bar{\beta}_{kj} F_{n-j} \subset \sum_{j=0}^{k} \bar{\gamma}_j \nabla^j F_n, \tag{11}$$

where the subset relation ($\subset$) is defined as not necessarily proper, and we get another kind of implicit interval methods corresponding to the conventional formula (7), namely

$$Y_n = Y_{n-1} + h\overline{\beta}_{k0}F_n + h\sum_{j=1}^{k}\overline{\beta}_{kj}F_{n-j} +$$

$$+ h^{k+2}\overline{\gamma}_{k+1}\overline{\Psi}\left(T_n + [-kh, 0],\ Y_n + [-kh, 0]\,F\left(\Delta_t, \Delta_y\right)\right), \quad (12)$$

$$n = k, k+1, ..., m,$$

where $h = \xi / m, t_i = ih \in T_i, i = 0,1,..., m,$ and $\overline{\beta}_{kj}$, $j = 0,1,...,k$ are given by (6). In particular for a given k from (12) we get the following methods:

- $k = 1$

$$Y_n = Y_{n-1} + \frac{h}{2}\left(F_n + F_{n-1}\right) +$$

$$-\frac{h^3}{12}\overline{\Psi}\left(T_n + [-h,0],\ Y_n + [-h,0]\,F\left(\Delta_t, \Delta_y\right)\right),$$

- $k = 2$

$$Y_n = Y_{n-1} + \frac{h}{12}\left(5F_n + 8F_{n-1} - F_{n-2}\right) +$$

$$-\frac{h^4}{24}\overline{\Psi}\left(T_n + [-2h,0], Y_n + [-2h,0]F\left(\Delta_t, \Delta_y\right)\right),$$

- $k = 3$

$$Y_n = Y_{n-1} + \frac{h}{24}\left(9F_n + 19F_{n-1} - 5F_{n-2} + F_{n-3}\right) +$$

$$-\frac{19}{720}h^5\overline{\Psi}\left(T_n + [-3h,0], Y_n + [-3h,0]F\left(\Delta_t, \Delta_y\right)\right).$$

If we denote by Y_n^1 the interval-solutions obtained from the formula (8) (or (9)), *i.e.* from the formula with backward interval differences, and by Y_n^2 the interval-solutions obtained from (12), then, we can prove

Theorem 1. $Y_n^2 \subset Y_n^1$,
which means that the second kind of implicit interval formulas gives the interval-solution with a smaller width, *i.e.* it is better. The proof of the Theorem (1) follows immediately from (11).

Let us note that (8) (or (9)) and (12) are nonlinear interval equations with respect to Y_n, $n = k, k + 1, ..., m$. It implies that in each step of implicit interval methods we have to solve an interval equation of the form

$$Y = G(T, Y),$$

where

$$T \in I\left(\Delta_t\right) \subset I\left(\mathbf{R}\right), \quad Y = \left(Y_1, Y_2, ..., Y_N\right)^T \in I\left(\Delta_y\right) \subset I\left(\mathbf{R}^N\right),$$

$$G : I\left(\Delta_t\right) \times I\left(\Delta_y\right) \to I\left(\mathbf{R}^N\right).$$

If we assume that the function G is a contracting mapping, then the well-known fixed-point theorem implies that the iteration process

$$Y^{(l+1)} = G\left(T, Y^{(l)}\right), \quad l = 0, 1, ..., \quad (13)$$

is convergent to Y^*, *i.e.* $\lim_{l\to\infty} Y^{(l)} = Y^*$, for an arbitrary choice of $Y^{(0)} \in I(\Delta_y)$.

For the interval methods of Adams-Moulton type given by (8) (or (9)), the iteration process (13) is of the form

$$Y_n^{(l+1)} = Y_{n-1} + h\sum_{j=0}^{k}\overline{\gamma}_j F\left(T_n, Y_n^{(l)}\right) +$$

$$+ h\sum_{j=1}^{k}\overline{\gamma}_j\sum_{m=1}^{j}(-1)^m\binom{j}{m}F\left(T_{n-m}, Y_{n-m}\right) + \quad (14)$$

$$+ h^{k+2}\overline{\gamma}_{k+1}\overline{\Psi}\left(T_n + [-kh, 0],\ Y_n^{(l)} +\right.$$

$$\left.+ [-kh, 0]\,F\left(\Delta_t, \Delta_y\right)\right), \ l = 0,\ 1, ...,\ n = k, k+1, ..., m,$$

and for the methods (12) of the form

$$Y_n^{(l+1)} = Y_{n-1} + h\overline{\beta}_{k0}F\left(T_n, Y_n^{(l)}\right) +$$

$$+ h\sum_{j=1}^{k}\overline{\beta}_{kj}F\left(T_{n-j}, Y_{n-j}\right) + \quad (15)$$

$$+ h^{k+2}\overline{\gamma}_{k+1}\overline{\Psi}\left(T_n + [-kh, 0],\right.$$

$$\left.Y_n^{(l)} + [-kh, 0]\,F\left(\Delta_t, \Delta_y\right)\right), \ l = 0, 1, ...,\ n = k, k+1, ..., m.$$

In (14) and (15) we usually choose $Y_n^{(0)} = Y_{n-1}$.

3. NUMERICAL EXAMPLE

Let us consider the motion of simple pendulum. The adequate equation is of the form

$$\ddot{\varphi} + u^2 \sin\varphi = 0, \quad (16)$$

where $\varphi = \varphi(t)$, $u = \sqrt{g / L}$, g is the gravitational acceleration at Earth's surface and L denotes the pendulum length.

Denoting $y_1 = \dot{\varphi}$, $y_2 = \varphi$, where $y_1 = y_1(t)$, $y_2 = y_2(t)$, we transform (16) with the initial conditions $\dot{\varphi}(0) = 0$, $\varphi(0) = \varphi_0$, into the following system of differential equations of the first order:

$$\dot{y}_1 = -u^2 \sin y_2, \quad \dot{y}_2 = y_1, \quad (17)$$

with the initial conditions

$$y_1(0) = 0, \quad y_2(0) = \varphi_0. \quad (18)$$

We have integrated (17) with (18) for $t = 1$ [s], where $\varphi_0 = \pi/60$ [rad], $g = 9.81$ [m/s^2], $L = 1$ [m]. We have used the interval methods of Adams-Moulton type (8) and (12) for the number of method steps $k = 1, 2, 3$, and for the stepsizes $h = 1E-3, 1E-4, 1E-5, 1E-6$.

In Figures 1 and 2 the comparison of the widths of two considered families is given. The widths of interval-solutions $Y_i(1)$, $i = 1, 2$, obtained with the IIAM methods (12) are always smaller than the ones that have been obtained by the IIAM methods (8) (or (9)) for the same

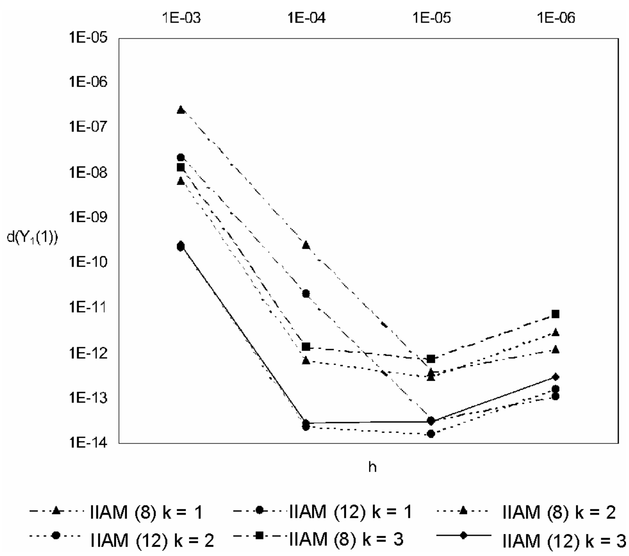

Fig. 1. Width of the interval-solution $Y_1(1)$ obtained by the IIAM methods (8) and (12) for (17) with (18), where $k = 1, 2, 3$, *vs.* the stepsize *h.* IIAM – the implicit interval methods of Adams-Moulton type

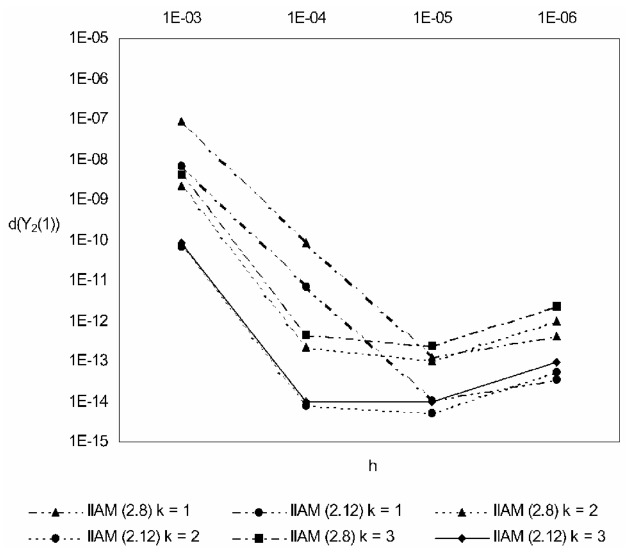

Fig. 2. Width of the interval-solution $Y_2(1)$ obtained by the IIAM methods (8) and (12) for (17) and (18), where $k = 1, 2, 3$, *vs.* the stepsize *h.* IIAM – the implicit interval methods of Adams-Moulton type

parameter k and stepsize h. The increase of parameter k for the same stepsize h contributes to the improvement of interval-solutions, *i.e*. decrease in their widths. The similar effect can be achieved if we reduce the stepsize h for the same value of parameter k. On the other side, let us notice that the widths of interval-solutions obtained by the IIAM methods (8) (or (9)) and (12) for $k = 3$ are somewhat worse that the once that have been obtained by the considered methods for $k = 2$. The similar effect can be observed for excessive decrease in the stepsize h. Hence, one can conclude that the optimal stepsize h and number of method steps k should be chosen for each particular interval method and each IVP.

4. SUMMARY

The main aim of our paper is to specify the precise form of two families of implicit interval methods of Adams-Moulton type. The reason for the existence of the second family has been explained in Sec. 2. The results of numerical experiment given is Sec. 3 have confirmed the thesis of Theorem 1. Furthermore, we have explained why the formulas (12) rather then (8) (or (9)) should be used in interval computations.

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