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COMPUTATIONAL METHODS FOR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract. Integral equations in space-time play very important role in mechanics and technology. Particular cases of these equations called mixed integral equations or Volterra-Fredholm integral equations arise in the heat conduction theory [4, 6] and the diffusion theory. Moreover, a current density in electromagnetism is determined by the Volterra-Fredholm integral equations [4]. Nonlinear counterparts of the equations studied in [1] are mathematical models of the spatio-temporal development of an epidemic (the spread of the disease in the given population). Some initial-boundary problems for a number of partial differential equations in physics are reducible to the considered integral equations [2- 3, 6], In this paper the general theory of these equations is used in the projection methods. Presented methods lead to a system of algebraic equations or to a system of Volterra integral equations. The convergence of studied algorithm is proved, the error estimate is established. The presented theory is illustrated by numerical examples.

1. GENERAL CONSIDERATIONS

The following integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_M k(x,t,y,s)\, u(y,s)\, dyds \tag{1}$$

in space-time is considered, where f is given function in domain $D = M \times [0, T]$ (M - a compact subset of n-dimensional Euclidean space) and u is unknown function in D; given kernel k is defined in domain $\Omega = \{(x,t,y,s) : x, y \in M, 0 \le s \le t \le T\}$. Integral equations of this type arise in various physical, mechanical and biological problems. The general theory for the considered equations in weighted spaces was presented in [3], Approximate solutions of Volterra-Fredholm integral equations were studied in papers [1-8]

The mixed integral equation (1) can be written in the operator form

$$u = f + Ku, \tag{2}$$

where

$$(Ku)(x,t) = \int_0^t \int_M k(x,t,y,s)\, u(y,s)\, dyds \tag{3}$$

is the integral operator in that a Volterra part plays the dominant role. On virtue this property we can prove the existence and uniqueness of solutions of the equation (1) in the spaces C and L^p ($p \ge 1$) [3].

Theorem 1

Let f and k be continuous functions on D and Ω respectively. Then the sequence $\{u_n\}$ defined by formula

$$u_n = f + Ku_{n-1}, \qquad n = 1, 2, 3, \ldots; \qquad u_0 = f$$

is uniformly convergent to unique solution of equation (1) given in the form

$$u(x, t) = f(x, t) + \int_0^t \int_M r(x, t, y, s) f(y, s)\, dyds,$$

where r is the resolvent kernel defined by the formula

$$r(x, t, y, s) = \sum_{n=1}^{\infty} k_{(n)}(x, t, y, s),$$

with iterated kernels

$$k_{(n)}(x, t, y, s) = \int_s^t \int_M k_{(1)}(x, t, u, v)\, k_{(n-1)}(u, v, y, s)\, dudv \quad \text{for } n = 2, 3, \ldots$$

$$k_{(1)}(x, t, y, s) = k(x, t, y, s).$$

Proof of this theorem is similar as in the case of Volterra integral equation

Remark 1

A solution of the Volterra-Fredholm integral equation can be presented in the operator form

$$u = (I - K)^{-1} f = \sum_{n=0}^{\infty} K^n f,$$

where

$$K^n = K\left(K^{n-1}\right), \quad n = 1, 2, \ldots, K^0 = I, K^1 = K.$$

2. GALERKIN TYPE METHOD

Classic al Galerkin method for integral equation (1) leads to approximate solution of the form

$$u_n(x, t) = \sum_{j=1}^{n} c_j \chi_j(x, t), \tag{4}$$

where $\{\chi_i\}$ is the orthogonal basis in the space $L^2(D)$. Because it is difficult to define such a system we propose the following formula

$$u_n(x, t) = \sum_{i,k=1}^{n} c_{i,k}\, \varphi_i(x)\, \psi_k(t), \tag{5}$$

where $\{\varphi_i\}$, $\{\psi_k\}$ are orthogonal bases in spaces $L^2(D)$ and $L^2[0, T]$, respectively.

Coefficients $c_{ik}(i, k = 1, 2, \ldots, n)$ are determined by the orthogonality condition in $L^2(D)$ of the form

$$(\varepsilon_n, \varphi_i \psi_k) = 0 \quad (i, k = 1, 2, \ldots, n), \tag{5}$$

where

$$\varepsilon_n = u_n - f - Ku_n,$$

is a deviation function.

In practice, we restrict our considerations to the orthonormal basis. Then we get the following system of linear algebraic equations

$$c_{ik} = f_{ik} + \sum_{j,m=1}^{n} c_{jm} k_{jm,ik} \qquad (i, k = 1, 2, \ldots, n) \tag{6}$$

where

$$f_{ik} = \int_0^T \int_M f(x, t)\, \varphi_i(x)\, \psi_k(t)\, dxdt, \tag{7}$$

$$k_{jm,ik} = \int_0^T \int_M \left[\int_0^t \int_M k(x, t, y, s)\, \varphi_j(y)\, \psi_m(s)\, dyds \right] \varphi_i(x)\, \psi_k(t)\, dxdt. \tag{8}$$

Theorem 2

Let $\{\varphi_i\}$, $\{\psi_k\}$, are orthonormal complete systems in the spaces $L^2(M)$ and $L^2[0, T]$, respectively. If $f \in L^2(D)$ and $k \in L^2(\Omega)$, then the system (6) is uniquely solvable and the sequence defined by the formula (5) converges to the unique solution of the equation (1) in the space $L^2(D)$.

The proof is similar as in the case of the Fredholm integral equation and it is based on the Fourier series theory.

Example 1

$$u(x, t) = x^2 \left(e^{-t} - \frac{2}{3} t^3 \right) + \int_0^t \int_{-1}^1 x^2 t^2 e^s u(y, s)\, dyds \qquad t \in \langle 0, 1 \rangle$$

Relative errors ($n = 6$)

$x\backslash t$	0.0	0.2	0.4	0.6	0.8	1.0
-1.0	.9619e-6	.2089e-6	.1250e-6	.1273e-6	.3487e-6	.2317e-5
-0.6	.9625e-6	.2093e-6	.1247e-6	.1270e-6	.3480e-6	.2315e-5
-0.2	.9685e-6	.2146e-6	.1208e-6	.1266e-6	.3432e-6	.2300e-5
0.2	.9685e-6	.2146e-6	.1208e-6	.1266e-6	.3432e-6	.2300e-5
0.6	.9625e-6	.2093e-6	.1247e-6	.1270e-6	.3480e-6	.2315e-5
1.0	.9619e-6	.2089e-6	.1250e-6	.1273e-6	.3487e-6	.2317e-5

Dependence of average relative error of a number (n) of basis functions

n	4	5	6	7
Relative errors	.337e-3	.163e-4	.665e-6	.328e-7

Example 2

$$u(x,t) = e^{x}t^{2} - \frac{2}{3}x^{2}t^{3} + \int_{0}^{t}\int_{-1}^{1} e^{-y}x^{2}\, u(y,s)\, dyds \qquad t \in \langle 0, 1\rangle$$

The average relative error dependence on the number (n) of basis functions

n	4	5	6	7
Relative errors	.506e-2	.549e-3	.615e-5	.344e-8

3. GALERKIN-FOURIER TYPE METHOD (GF - METHOD)

In this section we propose a projection method for the equation (1) leading to solve a system of Volterra linear integral equations. Numerical solution of (1) we search in the form

$$u_{n}(x,t) = \sum_{j=1}^{n} a_{j}(t)\, \varphi_{j}(x) \tag{9}$$

for $(x, t) \in D$, $D = M \times [0, T]$, where:
$\{\varphi_j\}$ is an orthonormal and complete basis in w $L^2(M)$;
$\{a_j\}$ is a solution to a system of the following Volterra integral equations

$$a_{j}(t) = f_{j}(t) + \sum_{k=1}^{n} \int_{0}^{t} k_{jk}(t,s)\, a_{k}(s)\, ds, \qquad j = 1, 2, \ldots, n \tag{10}$$

with

$$f_{j}(t) = \int_{M} f(x,t)\, \varphi_{j}(x)\, dx, \qquad j = 1, 2, \ldots, n \tag{11}$$

$$k_{jk}(t,s) = \int_{M}\int_{M} k(x,t,y,s)\, \varphi_{j}(x)\, \varphi_{k}(y)\, dydx \qquad k,j = 1, 2, \ldots, n. \tag{12}$$

Lemma 1

If $f \in L^2(D)$ and $k \in L^2(\Omega)$, $\Omega = \{(x, t, y, s) : 0 \le s \le t \le T; x, y \in M\}$, then function (9) is a unique solution in the space $L^2(D)$ of the equation

$$u_{n}(x,t) = f_{n}(x,t) + \int_{0}^{t}\int_{M} k_{n}(x,t,y,s)\, u_{n}(y,s)\, dyds, \tag{13}$$

with

$$f_n(x,t) = \sum_{k=1}^{n} f_k(t)\,\varphi_k(x), \tag{14}$$

$$k_n(x,t,y,s) = \sum_{j=1}^{n}\sum_{k=1}^{n} k_{jk}(t,s)\,\varphi_j(x)\,\varphi_k(y). \tag{15}$$

Proof. Putting (14) and (15) in (13) and using linear independence of system $\{\varphi_j\}$ we get

$$u_n(x\,t) = \sum_{k=1}^{n} u_k(t)\,\varphi_k(x), \tag{16}$$

where

$$u_k(t) = f_k(t) + \sum_{j=1}^{n} \int_0^t \int_M k_{jk}(t,s)\,\varphi_j(y)\,u_n(y,s)\,dyds, \qquad k = 1, 2, ..., n. \tag{17}$$

By orthonormality of $\{\varphi_k\}$ from (16) and (17) we obtain the Volterra system of integral equations

$$u_k(t) = f_k(t) + \sum_{j=1}^{n} \int_0^t k_{jk}(t,s)\,u_j(s)\,ds, \qquad k = 1, 2, ..., n. \tag{18}$$

It follows from the assumptions above and the Volterra theory, that this system has a unique solution $\{u_j\}$ in the space $L^2[0, T]$ such that $u_j(t) = aft)$ for every $j=1,2,\ldots, n$.

The equation (13) can be rewritten in the operator form

$$u_n = f_n + K_n u_n, \tag{19}$$

where K_n is the Volterra-Fredholm integral operator of the form (3) determined by the kernel k_n defined by the formula (15).

Theorem 3

If $f \in L^2(D)$ and $k \in L^2(\Omega)$, then the sequence $\{u_n\}$ defined by the formula (9) converges in the space $L^2(D)$ to the unique solution of the equation (1) and the estimated error

$$\|u_n - u\|_{L^2(D)} \le \frac{c}{1 - c\delta_n}\left[\|f - f_n\|_{L^2(D)} + \|u\|_{L^2(D)}\,\delta_n\right] \tag{20}$$

holds with

$$c = \|(I - K)^{-1}\| \qquad \text{and} \qquad \delta_n = \|k_n - k\|_{L^2(\Omega)}$$

($\|.\|$ is the operator norm).

Proof. Subtracting (19) and (2) we get

$$u_n - u = f_n - f + (K_n - K)\, u_n + K(u_n - u)\,.$$

Hence

$$\| u_n - u \|_{L^2(D)} \le \| (I - K)^{-1} \| \left[\| f_n - f \|_{L^2(D)} + \| k_n - k \|_{L^2(\Omega)} \| u_n \|_{L^2(D)} \right]. \tag{21}$$

Putting $p = 2$ in the theorem 2 and remark 1 of the reference [3] we obtain the following estimate

$$c = \| (I - K)^{-1} \| \le \sum_{j=0}^{\infty} \frac{\| k \|^j_{L^2(\Omega)}}{\sqrt{j!}}\,,$$

where

$$\| k \|_{L^2(\Omega)} = \left\{ \int_0^T \int_M \left[\int_0^t \int_M k^2 (x, t, y, s)\, dyds \right] dxdt \right\}^{\frac{1}{2}}.$$

Using the theory of Fourier series and properties of the Lebesgue'a integral [7] we have

$$F_n^2 (t) = \| f_n (\cdot, t) \|^2_{L^2(M)} \to 0 \qquad \text{for every} \qquad t \in [0, T]$$

and

$$\| F_n \|_{L^2[0, T]} = \left[\int_0^T \| f_n (\cdot, t) - f(\cdot, t) \|^2_{L^2(M)}\, dt \right]^{\frac{1}{2}} = \| f_n - f \|_{L^2(D)} \underset{n \to \infty}{\longrightarrow} 0\,.$$

Similarly

$$\| k_n (\cdot, t, \cdot, s) - k(\cdot, t, \cdot, s) \|^2_{L^2(M) \times L^2(M)} \to 0 \qquad \text{for every} \quad 0 \le s \le t \le T$$

and

$$\delta_n = \| k_n - k \|_{L^2(\Omega)} \underset{n \to \infty}{\longrightarrow} 0\,.$$

Then from (21) we get

$$\| u_n - u \|_{L^2(D)} \underset{n \to \infty}{\longrightarrow} 0\,.$$

To obtain estimate (20) let us notice

$$\| u_n \|_{L^2(D)} \le \| u_n - u \|_{L^2(D)} + \| u \|_{L^2(D)}\,,$$

where

$$u = (I - K)^{-1} f$$

is a unique solution of equation (1). By the above considerations we have

$$\|u_n - u\|_{L^2(D)} \le c\left[\|f_n - f\|_{L^2(D)} + \delta_n \|u_n\|_{L^2(D)}\right] \le c\,\|f_n - f\|_{L^2(D)} + c\delta_n \|u_n - u\|_{L^2(D)} + c\delta_n \|u\|_{L^2(D)}.$$

From the above we get the estimate (20).

4. IMPLEMENTATION OF THE GF METHOD IN MAPLE V

Let us consider the following integral equation of the Volterra-Fredholm type

$$u(x,t) = f(x,t) + \int_0^t \int_{-1}^1 k(x,t,y,s)\,u(y,s)\,dyds, \quad t \in [0,1] \tag{22}$$

```
restart;
  with(orthopoly);
```

```
[G, H, L, P, T, U]
```

```
  with(linalg):
Warning, new definition for norm
Warning, new definition for trace
un:=(k,x)->P(k-1,x)*((2*k-1)/2)^(1/2);
```

$$un := (k,x) \to P(k-1,x)\sqrt{k-\frac{1}{2}}$$

```
  wsp:=proc(l,d,h)
  local a,ind;
  a:=1:
  for ind from 0 to l do
  if ind<>d then
  a:=a*(r-ind)/(d-ind):
  fi od;
  h*int(a,r=0..l);
  end:

  fj:=(j,t)->int(f(x,t)*un(j,x),x=-1..1):
  kjk:=(j,k,t,s)->int(int(k(x,t,y,s)*un(j,x)*un(k,y),x=-
1..1),y=-1..1):
  wekG:=proc(n,k,h)
  local i;
  global G;
  G:=vector(n);
  for i from 1 to n do
    if k=0 then G[i]:=limit(fj(i,t),t=0)
      else G[i]:=fj(i,k*h):fi:
    od;op(G)end:
```

```
macNe:=proc(n,k,h)
local i,j,a;
global Ne, ej;
Ne:=matrix(n,n);
ej:=matrix(n,n);
for i from 1 to n do
   for j from 1 to n do
   if i=j then
   ej[i,j]:=1;
else ej[i,j]:=0;fi;
   Ne[i,j]:=kjk(i,j,k*h,k*h); od;od;
a:=wsp(k,k,h);
Ne:=matadd(ej,Ne,1,-a);
for i from 1 to n do
for j from 1 to n do
Ne[i,j]:=evalf(Ne[i,j]); od; od;
end:

macN:=proc(n,k,j,h)
local i,ii;
global Ni;
Ni:=matrix(n,n);
   for i from 1 to n do
    for ii from 1 to n do

if j=0 then Ni[i,ii]:=evalf(wsp(k,j,h)*limit(kjk(i,ii,k*h,s),s=0))
          else Ni[i,ii]:=evalf(wsp(k,j,h)*kjk(i,ii,k*h,j*h)); fi;
od; od;
end:

wynA:=proc(n,k,h)
local i,j;
global A,V,wA,Z,pomw,pom1w;
pomw:=vector(n);
pom1w:=vector(n);
V:=vector(n);
wA:=vector(n);
Z:=vector(n);
if k=0 then
   for i from 1 to n do
   A[i,1]:=gj(i,0); od;fi;
if k>0 then
for i from 1 to n do
     V[i]:=0 od;
   for j from 0 to k-1 do
      for i from 1 to n do
        wA[i]:=A[i,j+1]; od;
```

```
      mack(n,k,j,h):
      pomw:=multiply(Ni,wA);
      V:=matadd(pomw,V); od;
   wekG(n,k,h):
   pom1w:=matadd(G,V);
   macNe(n,k,h):
   Z:=linsolve(Ne,pom1w);
for i from 1 to n do
A[i,k+1]:=evalf(Z[i]); od;
fi;
end:

macA:=proc(n,h)
local k;
global A;
A:=matrix(n,krok+1);
for k from 0 to krok do
wynA(n,k,h);od;
   end
```

5. NUMERICAL EXPERIMENTS

The presented above theory is illustrated by the Legendre polynomials $\{P_j\}$ in the orthonormalized form

$$\varphi_j(x) = \frac{P_j(x)}{\|P_j\|_{L^2(-1,1)}}$$

forming a complete system in $L^2_{(-1,1)}$.

Functions a_j *(j*= 1,2,..., *n)* determining the approximate solution (9) of the equation (1) are calculated by the system of Volterra integral equations (10) using the Newton-Cotes quadrature.

Example 3

Consider the integral equation

$$u(x,t) = x^2\left(e^{-t} - \frac{2}{3}t^3\right) + \int_0^t \int_{-1}^1 x^2 t^2 e^s u(y,s)\,dyds \qquad t \in \langle 0,1\rangle$$

with an exact solution

$$u(x,t) = e^{-t}x^2.$$

The Galerkin-Fourier method restricted to following basis functions $\{\varphi_j\}_{j=1,2,3}$ leads to solve the system of $n = 3$ integral equations of the Volterra type.

The relative errors with the step of quadrature formula $h = 0.1$ are given below:

$x\backslash t$	−1	−0.6	−0.2	0.2	0.6	1
0.2	$.1221 \cdot 10^{-9}$	$.6785 \cdot 10^{-10}$	$.3053 \cdot 10^{-8}$	$.3053 \cdot 10^{-8}$	$.6785 \cdot 10^{-10}$	$.1221 \cdot 10^{-9}$
0.4	$.1492 \cdot 10^{-9}$	$.7459 \cdot 10^{-9}$	0	0	$.7459 \cdot 10^{-9}$	$.1492 \cdot 10^{-9}$
0.6	$.7288 \cdot 10^{-9}$	$.7592 \cdot 10^{-9}$	0	0	$.7592 \cdot 10^{-9}$	$.7288 \cdot 10^{-9}$
0.8	0	$.5563 \cdot 10^{-9}$	$.5564 \cdot 10^{-8}$	$.5563 \cdot 10^{-8}$	$.5564 \cdot 10^{-9}$	0
1	$.5436 \cdot 10^{-9}$	$.1034 \cdot 10^{-8}$	$.6795 \cdot 10^{-8}$	$.6795 \cdot 10^{-8}$	$.1034 \cdot 10^{-8}$	$.5436 \cdot 10^{-9}$

Experiments for $n = 4$, $n = 5$ and $n = 6$ give similar errors.

Example 4

As an approximate solution of the integral equation

$$u(x,t) = e^{xt} - 2x^2 t + \int_0^t \int_{-1}^{1} x^2 e^{-sy} u(y,s)\,dyds$$

with the exact solution

$$u(x,t) = e^{xt}$$

we propose for $n = 4$ and $n = 6$ basis functions with $h = 0.1$.

The tables below give relative errors dependence on n- the number of basis functions

$n = 4$

$x\backslash t$	0.2	0.4	0.6	0.8	1.0
−1.0	.2833e−4	.0035	.00214	.0820	.02435
−0.6	.1020e−4	.00012	.00017	.0255	.00707
−0.2	.1131e−5	.5804e−4	.00029	.00096	.00240
0.2	.5585e−5	.5567e−4	.00028	.00088	.00216
0.6	.3000e−4	.8005e−4	.00036	.00104	.00231
1.0	.6409e−5	.00017	.00073	.00197	.00411

$n = 6$

$x\backslash t$	0.2	0.4	0.6	0.8	1.0
−1.0	.0028	.00107	.00011	.5638e−4	.00025
−0.6	.0010	.00014	.9515e−5	.6368e−5	.2666e−4
−0.2	.5235e−4	.00024	.1871e−4	.1348e−5	.6988e−5
0.2	.5284e−4	.00020	.1647e−4	.2333e−5	.8495e−5
0.6	.7976e−4	.8283	.7821e−5	.3284e−5	.1078e−4
1.0	.00018	.00047	.3073e−4	.1190e−4	.3961e−4

In the following examples the error dependence on the number of the basis functions, n, and on a step of the quadrature formula, h, are presented.

Example 5

$$u(x,t) = e^x t^2 - \frac{2}{3}x^2 t^3 + \int_0^t \int_{-1}^1 x^2 e^{-y}\, u(y,s)\, dyds$$

$n = 7,\ h = 0.1$

$x\backslash t$	2	0.4	0.6	0.8	1
−1.0	.002951504540	.0009316754594	.0005217387064	.0003011446842	.0003494498724
−0.8	.001471686767	.0004491672421	.0002470762282	.0001888029023	.0001057247083
−0.6	.0007038132108	.0002215284454	.0001246720567	.00007907573039	.00007494071325
−0.4	.0002628464290	.00008236104231	.00004531180795	.00002096846684	.00003459125934
−0.2	.00004535992933	.00001150265867	$.4844021350 \cdot 10^{-5}$	$.5303752423 \cdot 10^{-5}$	$.1050509214 \cdot 10^{-5}$
0.0	$.894174894 \cdot 10^{-5}$	$.395049370 \cdot 10^{-5}$	$.231927762 \cdot 10^{-5}$	$.595781079 \cdot 10^{-5}$	$.897942294 \cdot 10^{-5}$
0.2	.00003480231055	.00001196486759	$.7585095423 \cdot 10^{-5}$	$.7079740259 \cdot 10^{-5}$	$.3106367965 \cdot 10^{-5}$
0.4	.0001187963372	.00003759948178	.00002101366789	$.9492307633 \cdot 10^{-5}$	.00001581455058
0.6	.0002090352867	.00006382763731	.00003462899861	.00002117429978	.00001982262718
0.8	.0002982311963	.00009189269894	.00005103222412	.00003985061584	.00002286854186
1.0	.0003949166083	.0001213382348	.00006597568595	.00003491359138	.00004189697059

$n = 7,\ ,h = 0.05$

$x\backslash t$	2	0.4	0.6	0.8	1
−1.0	.0002885010120	.0001139410232	.00006799026720	.00002046033564	.0001577615308
−0.8	.00007630515338	.00002068412536	$.9316806904 \cdot 10^{-5}$	.00002028007483	$.5287604116 \cdot 10^{-5}$
−0.6	.00006118947740	.00002419606550	.00001517449982	$.1458964314 \cdot 10^{-5}$	.00002869052770
−0.4	.00002900805438	.00001055532387	$.5467117942 \cdot 10^{-5}$	$.7280818877 \cdot 10^{-5}$	.00001776675876
−0.2	$.2502936280 \cdot 10^{-5}$	$.3195244098 \cdot 10^{-5}$	$.3312336631 \cdot 10^{-5}$	$.4851241178 \cdot 10^{-5}$	$.4488635838 \cdot 10^{-5}$
0.0	$.894204894 \cdot 10^{-5}$	$.395179370 \cdot 10^{-5}$	$.232097762 \cdot 10^{-5}$	$.595061079 \cdot 10^{-5}$	$.897332294 \cdot 10^{-5}$
0.2	$.2718872275 \cdot 10^{-5}$	$.2112568726 \cdot 10^{-5}$	$.2117725166 \cdot 10^{-5}$	$.3199340269 \cdot 10^{-5}$	$.8017227680 \cdot 10^{-5}$
0.4	.00001372598260	$.5335092700 \cdot 10^{-5}$	$.3110294603 \cdot 10^{-5}$	$.3200914650 \cdot 10^{-5}$	$.8254815171 \cdot 10^{-5}$
0.6	.00001548074193	$.4392266667 \cdot 10^{-5}$	$.1648971310 \cdot 10^{-5}$	$.2203420906 \cdot 10^{-5}$	$.5892338990 \cdot 10^{-5}$
0.8	.00001650850702	$.5383449634 \cdot 10^{-5}$	$.3029424842 \cdot 10^{-5}$	$.5826443772 \cdot 10^{-5}$	$.2590640238 \cdot 10^{-5}$
1.0	.00003451827128	.00001066991321	$.4567512395 \cdot 10^{-5}$	$.8610915097 \cdot 10^{-5}$	.00001595477458

$n = 6,\ h = 0.1$

$x\backslash t$	2	0.4	0.6	0.8	1
-1.0	.002604990238	.0006284749356	.0002329752778	.00008457112181	$.2932852123 \cdot 10^{-5}$
-0.8	.001582839876	.0005464263584	.0003397031542	.0002582730042	.0002168757585
-0.6	.0006638395158	.0001865514741	.00009136011387	.00005409212250	.00003496520962
-0.4	.0002071956364	.00003366659415	$.1064235831 \cdot 10^{-5}$	.00001381306874	.00002106109221
-0.2	.00005790561650	.00002248013113	.00001529844682	.00001314513134	.00001149398708
0.0	.000030894341	.000030906185	.000030876664	.000030855667	.000030855667
0.2	.00004321193615	.00001932328743	.00001459290639	.00001233597374	.00001151519530
0.4	.00009379082420	.00001571965583	$.1755681782 \cdot 10^{-5}$	$.6136043716 \cdot 10^{-5}$	$.9191662880 \cdot 10^{-5}$
0.6	.0001969954452	.00005329277601	.00002459564492	.00001364938170	$.7782236866 \cdot 10^{-5}$
0.8	.0003206726218	.0001115289759	.00006973327798	.00005387638752	.00004530955186
1.0	.0003480209970	.00008030450600	.00002689580554	$5603546948 \cdot 10^{-5}$	$.4999008513 \cdot 10^{-5}$

$n = 6,\ h = 0.05$

$x\backslash t$	2	0,4	0,6	0,8	1
-1.0	.00005801396968	.0001892595008	.0002207724063	.0002370292261	.0001887848469
-0.8	.0001874582622	.0001179429634	.0001019442894	.00008975191536	.0001164250785
-0.6	.00002121579588	.00001078090582	.00001813691663	.00002352450122	.00001128925790
-0.4	.00002664285010	.00003813921755	.00004090863578	.00004206305375	.00003788518999
-0.2	.00001004265929	$.7782304700 \cdot 10^{-5}$	$.7142428123 \cdot 10^{-5}$	$.7355300578 \cdot 10^{-5}$	$.8058058983 \cdot 10^{-5}$
0.0	.000030894141	.000030904885	.000030875864	.000030847667	.000030864167
0.2	.00001112843647	$.9471039744 \cdot 10^{-5}$	$.9125763559 \cdot 10^{-5}$	$.8454934113 \cdot 10^{-5}$	$.9212023820 \cdot 10^{-5}$
0.4	.00001127958065	.00001654477515	.00001772767477	.00001882958022	.00001675121731
0.6	$.3440900423 \cdot 10^{-5}$	$.6142594648 \cdot 10^{-5}$	$.8384229942 \cdot 10^{-5}$	$.9728296119 \cdot 10^{-5}$	$.6149341031 \cdot 10^{-5}$
0.8	.00003894993265	.00002501967038	.00002173059103	.00001985256647	.00002502890933
1.0	.00001237743192	.00003036381558	.00003451226581	.00003792032722	.00003094517762

Example 6

$n = 5,\ h = 0.1$

$x\backslash t$	2	0.4	0.6	0.8	1
-1.0	.002343885871	.0024207066374	.002416741450	.002402087346	.002403449668
-0.8	.001219091966	.001153895062	.001158566256	.001169497521	.001168305460
-0.6	.0005981874547	.0005445509830	.0005485517766	.0005573862628	.0005564015464
-0.4	.0001834771533	.0002270142114	.0002239587437	.0002165489282	.0002173411938
-0.2	.0002425828288	.0002778713676	.0002755868489	.0002693541824	.0002699903908
0.0	.0001143370838	.00008556305180	.00008735533923	.00009251034134	.000091995333
0.2	.0003874699704	.0003638093062	.0003653302282	.0003695006930	.0003690765074
0.4	.0003164039567	.0002968338064	.0002981931269	.0003015138084	.0003011605999
0.6	$.1333389750 \cdot 10^{-5}$	.00001749322625	.00001629760012	.00001364321665	.00001393727840
0.8	.0001502976117	.0001634605285	.0001625145636	.0001603074217	.0001605471728
1.0	.0004973996422	.0004870116182	.0004875667785	.0004895604622	.0004893743854

$n = 5, h = 0.05$

$x\backslash t$	2	0.4	0.6	0.8	1
−1.0	.002724038230	.002724449388	.002726571021	.002725269276	.002635240278
−0.8	.0009078495393	.0009052113166	.0009048992582	.0009048985352	.0009785313593
−0.6	.0003433637081	.0003409459530	.0003408668041	.0003407509362	.0004010276542
−0.4	.0003921091913	.0003937119110	.0003939968176	.0003939149322	.0003445505776
−0.2	.0004133962943	.0004143519008	.0004148022491	.0004145691844	.0003741406254
0.0	.00002551315345	.00002617775784	.00002662459027	.00002638164668	$.6724333 \cdot 10^{-5}$
0.2	.0002729702802	.0002723236690	.0002720113546	.0002721601662	.0002992625173
0.4	.0002226595392	.0002219317016	.0002217900954	.0002218181254	.0002440017393
0.6	.00007808482727	.00007881788282	.00007885111172	.00007889252310	.00006073499543
0.8	.0002131363740	.0002136689106	.0002137290472	.0002137290356	.0001988619029
1.0	.0004459516151	.0004459044718	.0004456362241	.0004458225440	.0004580049376

$n = 6,\ h = 0.1$

$x\backslash t$	2	0.4	0.6	0.8	1
−1.0	.0006681236336	.0005728582112	.0005857252545	.0005877641316	.0005931072997
−0.8	.0002339453603	.0001748791371	.0001767390503	.0001918761365	.0001885199628
−0.6	.0002900594076	.0002383758272	.0002415336087	.0002517123882	.0002500613044
−0.4	.0002639255834	.0002176829060	.0002220957504	.0002276595554	.0002278734767
−0.2	.0001736311681	.0001358203922	.0001393623016	.0001438733842	.0001441716672
0.0	.0001143626488	.00008561987526	.00008744445973	.00009262495074	.000092119333
0.2	.0001085152463	.00008659647292	.00008732742593	.00009269364010	.00009165842272
0.4	.0001154072219	.00009709469108	.00009788740171	.0001020717206	.0001012789926
0.6	.00009150105526	.00007478732926	.00007627231576	.00007854978302	.00007846673466
0.8	.00004862303190	.00003425044264	.00003579301950	.00003717392628	.00003737954102
1.0	.00008978729217	.00008191851548	.00008129262232	.00008501239032	.00008392573340

$n = 6, h = 0.05$

$x\backslash t$	2	0.4	0.6	0.8	1
−1.0	.0002879712750	.0002691142740	.0002758933314	.0002645993500	.0003611481565
−0.8	.00007729706655	.00007380536532	.00007692987343	.00007208808487	$.1392121252 \cdot 10^{-5}$
−0.6	.00003523566100	.00003477017734	.00003384705971	.00003508855698	.00009457444082
−0.4	.00005529354542	.00005098469886	.0000520568579	.00005030296280	.0001005715998
−0.2	$.2817702621 \cdot 10^{-5}$	$.6605565040 \cdot 10^{-6}$	$.1458445038 \cdot 10^{-6}$	$.1333912417 \cdot 10^{-5}$	.00003994570569
0.0	.00002548758852	.00002612127462	.00002653633502	.00002626072852	$.6786333 \cdot 10^{-5}$
0.2	$.5984443762 \cdot 10^{-5}$	$.4889442888 \cdot 10^{-5}$	$.5992156005 \cdot 10^{-5}$	$.4641721588 \cdot 10^{-5}$	.00002179367136
0.4	.00002166280446	.00002219235834	.00002148379014	.00002238026658	.00004407857216
0.6	.00001474961774	.00001346248596	.00001371832929	.00001330393889	.00003163499131
0.8	.00001421573036	.00001595809250	.00001542185295	.00001624485284	$.9630474879 \cdot 10^{-6}$
1.0	.00003833926503	.00004081124370	.00003936161313	.00004127679308	.00005253347704

6. CONCLUSIONS

Comparing the examples 1 and 3 we notice that for $n = 3$ the method of the Galerkin-Fourier type gives better results than the Galerkin type method for $n = 4, 5, 6, 7$. The G-F method is dependent on the precision of solving the system of the Volterra integral equations; in the example 2 better results are obtained than in the example 5. Increasing the number of the basis functions, n, and the step of the quadrature formula, h, gives results of increasing accuracy.

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